AMSC/CMSC 460 Computational Methods

Exam 3, Thursday, April 30, 2015 **Solution**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.

   (a) ______ A consistent scheme has a local accuracy of at least first order.

   Solution: False. The order of accuracy can be less than 1.

   (b) ______ Runge-Kutta methods are one-step methods.

   Solution: True.

   (c) ______ One can design an implicit fourth order scheme for solving ODEs which is A-stable.

   Solution: True. For example, implicit RK4 method.

   (d) ______ Matlab function `ode23s` uses implicit methods.

   Solution: True. To treat with stiffness system, A-stable methods have to be chosen, which needs to be implicit.

   (e) ______ If u is a weak solution of the boundary value problem of an elliptic equation, it is also a strong solution of the same equation.

   Solution: False. A weak solution might not be in $\mathcal{C}^2$, in which case it is not a strong solution.

2. The following Matlab code discribes *midpoint method* which solves the ODE $y' = f(x, y)$ with initial value $y(0) = y_0$.

```
for i = 1:N+1
    y(i+1) = y(i)+h*f(x(i)+1/2*h,y(i)+1/2*h*f(x(i),y(i)));
end
```

(a) (5 points) Write down the scheme. Is it explicit or implicit?

Solution: The scheme is explicit, and it reads

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right).$$

(b) (10 points) Express the truncation error $T_n(h)$. Prove that $T_n(h) = \mathcal{O}(h^2)$. What is the local order of accuracy?

Solution: The truncation error is given as

$$T_n(h) = \frac{y(x_{n+1}) - y(x_n)}{h} - f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right).$$

From the ODE, we get

$$\frac{y(x_{n+1}) - y(x_n)}{h} = f + \frac{h}{2}(f_x + f f_y) + \mathcal{O}(h^2).$$

Here we write $f = f(x_n, y_n)$ for simplicity.
From the scheme, we have

$$f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right) = f + \frac{h}{2} f_x + \frac{hf}{2} f_y + \mathcal{O}(h^2).$$

After cancellation of lower order terms, the truncation error is

$$T_n = \mathcal{O}(h^2).$$

Therefore, the method has second order accuracy.

(c) (10 points) Obtain the region of absolute stability of the scheme. Is the scheme A-stable?

To proceed, consider the initial value problem $y' = \lambda y$ with $y(0) = y_0$. The exact solution is $y(x) = y_0 e^{\lambda x}$. For $\lambda < 0$, $|y(x)|$ decays as x becomes larger. Find the rigion of $z = \lambda h$ in $\mathbb{R}$ such that the scheme is stable, namely $|y_{n+1}| < |y_n|$.

Solution: Plug in $f(x, y) = \lambda y$ to the scheme, we get

$$y_{n+1} = y_n + h\lambda\left(y_n + \frac{h}{2}\lambda y_n\right) = y_n(1 + z + \frac{z^2}{2}).$$

To ensure stability, we need z inside the region of absolute stability:

$$\left|1 - z + \frac{z^2}{2}\right| < 1.$$

For $z \in \mathbb{R}$, the condition is equivalent to $-2 < z < 0$. As the region does not contain the whole $\mathbb{R}^-$, the scheme is not A-stable.

(d) (5 points) State the mathematical definition for: the method converges at $x = 1$.

Solution: Given any $h = 1/N$, denote y_N as the solution after applying the scheme N times with stepsize h. Convergence means

$$\lim_{N\to\infty} y_N - y(1) = 0,$$

where $y(1)$ is the exact solution of the ODE at $x = 1$.

(e) (5 points) Does the method converge? What is the rate of convergence? (Just state the result. No need to prove.)

Solution: The method has second order convergence, if h is small enough so that z lies in the region of stability. (Accuracy + Stability = Convergence).

3. We modify the midpoint method as follows.

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h\ ,\ y_{n+1} - \frac{h}{2}f(x_{n+1}, y_{n+1})\right).$$

(a) (5 points) Write down the truncation error $T_n(h)$.

Solution:

$$T_n(h) = \frac{y(x_{n+1}) - y(x_n)}{h} - f\left(x_n + \frac{1}{2}h\ ,\ y(x_{n+1}) - \frac{h}{2}f(x_{n+1}, y(x_{n+1}))\right).$$

(b) (10 points) Check the local order of accuracy of the scheme. (What is the order of accuracy?)

Solution: From the ODE, we get

$$\frac{y(x_{n+1}) - y(x_n)}{h} = f + \frac{h}{2}(f_x + ff_y) + \mathcal{O}(h^2).$$

Here we write $f = f(x_n, y_n)$ for simplicity.
From the scheme, we have

$$\begin{aligned}
& f\left(x_n + \frac{h}{2}, y(x_{n+1}) - \frac{h}{2} f(x_n, y(x_{n+1}))\right) \\
= \ & f + \frac{h}{2} f_x + f_y \cdot \left[y(x_{n+1}) - \frac{h}{2} f(x_n, y(x_{n+1})) - y_n\right] + \mathcal{O}(h^2) \\
= \ & f + \frac{h}{2} f_x + f_y \cdot \left[hf + \mathcal{O}(h^2) - \frac{h}{2}\left(f + \mathcal{O}(h)\right)\right] + \mathcal{O}(h^2) \\
= \ & f + \frac{h}{2} f_x + f_y \cdot \frac{h}{2} f.
\end{aligned}$$

In the second equality, we again use the fact that $y(x_{n+1}) = y_n + hf + \mathcal{O}(h^2)$.
After cancellation of lower order terms, the truncation error is

$$T_n = \mathcal{O}(h^2).$$

Therefore, the method has second order accuracy.

(c) (10 points) Obtain the region of absolute stability of the scheme. Is the scheme A-stable?

Solution: Plug in $f(x, y) = \lambda y$ to the scheme, we get

$$y_{n+1} = y_n + h\lambda\left(y_{n+1} - \frac{h}{2}\lambda y_{n+1}\right).$$

Denote $z = \lambda h$. Reorganize the scheme in the explicit way:

$$\left(1 - z + \frac{z^2}{2}\right) y_{n+1} = y_n, \quad \Rightarrow \quad y_{n+1} = \left(1 - z + \frac{z^2}{2}\right)^{-1} y_n.$$

The region of absolute stability is given by

$$\left|1 - z + \frac{z^2}{2}\right| > 1.$$

For $z \in \mathbb{R}$, the condition is equivalent to $z \in (-\infty, 0) \cup (2, \infty)$. As $\mathbb{R}^-$ lies inside the region of absolute stability, the method is A-stable.

4. Consider the following boundary value problem of second order ODE:

$$\begin{cases} -\left[(x+1)u'(x)\right]' + u(x) = e^{-x} \\ u(0) = 0, \quad u(1) = 0 \end{cases}.$$

(a) (5 points) Write down the weak formulation of the problem.

Solution: The weak formulation is given as for all test function $v \in \mathbb{H}_0^1([0,1])$,

$$\int_0^1 \left[(x+1)u'(x)v'(x) + u(x)v(x)\right] dx = \int_0^1 e^{-x} v(x) dx.$$

(b) (20 points) We use space of linear splines to approximate the weak solution, with respect to equally distributed nodes $\{x_i\}_{i=0}^4$, where $x_i = i/4$. One can express the approximate solution $u^h = \sum_{i=1}^3 c_i \phi_i(x)$, where $\{\phi_i\}_{i=1}^3$ are hat functions. Set up the linear system which solves $\{c_i\}_{i=1}^3$:

$$(K+M)c = b.$$

Find and evaluate the stiffness matrix K and mass matrix M.

Solution: The system is a 3-by-3 linear system, where

$$K_{ij} = \int_0^1 (x+1)\phi_i'(x)\phi_j'(x)dx, \quad M_{ij} = \int_0^1 \phi_i(x)\phi_j(x)dx, \quad b_i = \int_0^1 e^{-x}\phi_i(x)dx.$$

Note that K and M are tridiagonal matrices, namely $K_{13} = K_{31} = M_{13} = M_{31} = 0$. To calculate K, we start with diagonal entries:

$$K_{ii} = \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} (x+1)dx = \frac{1}{h^2}\left(\frac{x^2}{2} + x\right)\Bigg|_{x_{i-1}}^{x_{i+1}}.$$

So, $K_{11} = 10, K_{22} = 12, K_{33} = 14$. For off-diagonal entries:

$$K_{i,i+1} = K_{i+1,i} = -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x+1)dx = -\frac{1}{h^2}\left(\frac{x^2}{2} + x\right)\Bigg|_{x_i}^{x_{i+1}}.$$

So, $K_{12} = K_{21} = -11/2$, $K_{23} = K_{32} = -13/2$. To conclude,

$$K = \begin{pmatrix} 10 & -11/2 & 0 \\ -11/2 & 12 & -13/2 \\ 0 & -13/2 & 14. \end{pmatrix}.$$

For the mass matrix M, clearly $M_{11} = M_{22} = M_{33}$, and $M_{12} = M_{21} = M_{23} = M_{32}$. We will only calculate M_{11} and M_{12}.

$$M_{11} = 2\int_0^{1/4} (4x)^2 dx = \frac{1}{6}, \quad M_{12} = \int_{1/4}^{1/2} (4x-1)(2-4x)dx = \frac{1}{24}.$$

We conclude

$$M = \begin{pmatrix} 1/6 & 1/24 & 0 \\ 1/24 & 1/6 & 1/24 \\ 0 & 1/24 & 1/6. \end{pmatrix}.$$