# AMSC/CMSC 460 Computational Methods

**Exam 3, Thursday, April 30, 2015** Name: ________________________

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.
   (a) ______ A consistent scheme has a local accuracy of at least first order.
   (b) ______ Runge-Kutta methods are one-step methods.
   (c) ______ One can design an implicit fourth order scheme for solving ODEs which is A-stable.
   (d) ______ Matlab function `ode23s` uses implicit methods.
   (e) ______ If $u$ is a weak solution of the boundary value problem of an elliptic equation, it is also a strong solution of the same equation.

2. The following Matlab code discribes *midpoint method* which solves the ODE $y' = f(x, y)$ with initial value $y(0) = y_0$.

```
for i = 1:N+1
    y(i+1) = y(i)+h*f(x(i)+1/2*h,y(i)+1/2*h*f(x(i),y(i)));
end
```

   (a) (5 points) Write down the scheme. Is it explicit or implicit?
   (b) (10 points) Express the truncation error $T_n(h)$. Prove that $T_n(h) = \mathcal{O}(h^2)$. What is the local order of accuracy?
   (c) (10 points) Obtain the region of absolute stability of the scheme. Is the scheme A-stable?
   To proceed, consider the initial value problem $y' = \lambda y$ with $y(0) = y_0$. The exact solution is $y(x) = y_0 e^{\lambda x}$. For $\lambda < 0$, $|y(x)|$ decays as $x$ becomes larger. Find the rigion of $z = \lambda h$ in $\mathbb{R}$ such that the scheme is stable, namely $|y_{n+1}| < |y_n|$.
   (d) (5 points) State the mathematical definition for: the method converges at $x = 1$.
   (e) (5 points) Does the method converge? What is the rate of convergence? (Just state the result. No need to prove.)

3. We modify the midpoint method as follows.

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h\ ,\ y_{n+1} - \frac{h}{2}f(x_{n+1}, y_{n+1})\right).$$

(a) (5 points) Write down the truncation error $T_n(h)$.

(b) (10 points) Check the local order of accuracy of the scheme. (What is the order of accuracy?)

(c) (10 points) Obtain the region of absolute stability of the scheme. Is the scheme A-stable?

4. Consider the following boundary value problem of second order ODE:

$$\begin{cases} -\big[(x+1)u'(x)\big]' + u(x) = e^{-x} \\ u(0) = 0, \quad u(1) = 0 \end{cases} .$$

(a) (5 points) Write down the weak formulation of the problem.

(b) (20 points) We use space of linear splines to approximate the weak solution, with respect to equally distributed nodes $\{x_i\}_{i=0}^4$, where $x_i = i/4$. One can express the approximate solution $u^h = \sum_{i=1}^3 c_i\phi_i(x)$, where $\{\phi_i\}_{i=1}^3$ are hat functions. Set up the linear system which solves $\{c_i\}_{i=1}^3$:

$$(K+M)c = b.$$

Find and evaluate the stiffness matrix $K$ and mass matrix $M$.