AMSC/CMSC 460 Computational Methods

Exam 2, Due Tuesday April 14, 2015 Name: ______________________

This exam is now take-home due to power outage on April 7. Please read carefully with the instructions below.

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. **Use no books, calculators, computers, internet, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. Use no more than 80 minutes to finish the exam.**

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.
   (a) ______ Lagrange interpolations on equally distributed nodes have the best perfomance (in the sense of minimizing L^∞ error) when the number of nodes n is large.
   (b) ______ A natural cubic spline is a $\mathcal{C}^2$ function.
   (c) ______ Newton-Cotes type integrations use equally spaced nodes.
   (d) ______ Suppose $s_1(x)$ is a linear interpolating spline of $f(x)$ on equally distributed nodes in $[a, b]$. Then, $\int_a^b s_1(x)dx$ defines a composite trapezoid rule to approximate $\int_a^b f(x)dx$.
   (e) ______ A Gauss quadrature with n nodes has higher algebraic accuracy than a Newton-Cotes type quadrature with n nodes.

2. Let $f(x) = (1+x)^{-1}$, for $x \in [0,1]$.

   (a) (8 points) Find a cubic polynomial $p_3(x)$ which interpolates f such that

   $$p_3(0) = f(0), \quad p_3'(0) = f'(0), \quad p_3''(0) = f''(0), \quad \text{and } p_3(1) = f(1).$$

   (b) (6 points) Obtain an error bound uniformly in $[0,1]$. Namely, find an upper bound of $\|f - p_3\|_{L^\infty([0,1])}$.

   (c) (3 points) $s_L(x)$ is the linear interpolating spline for f, on nodes $\{x_i\}_{i=0}^5$, where $x_i = i/5$. Let $\{\varphi_i(x)\}_{i=0}^5$ be hat functions with respect to the nodes. We can express $s_L(x) = \sum_{k=0}^{m} a_k \varphi_k(x)$. Find $a_0, \cdots, a_5$.

   (d) (6 points) (*) Find an error bound of $\|f - s_L\|_{L^\infty([0,1])}$.

3. (20 points) Let $f(x) = x^4$. Find the quadratic polynomial $p_2(x)$ which minimizes the following functional

$$\int_0^\infty (f(x) - p_2(x))^2 e^{-x} dx.$$

You do not have to simplify your answer.

Hint: To ease the computational load, you can use the following identity,

$$\int_0^\infty x^k e^{-x} dx = k!, \quad \textit{for all integers} \;\; k \geq 0.$$

4. Simpson's 3/8 rule states the following. Let $\{x_i\}_{i=0}^3$ be equally distributed nodes in $[a, b]$, namely $x_i = a + i(b - a)/3$. Then,

$$\int_a^b f(x)dx \approx I_3[f] := \frac{b-a}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right).$$

The corresponding error formula reads

$$E = \int_a^b f(x)dx - I_3[f] = -\frac{(b-a)^5}{6480} f^{(4)}(\xi), \quad \text{where } \ \xi \in (a, b).$$

(a) (5 points) What is the algebraic accuracy of the Simpson's 3/8 rule.

(b) (5 points) Write a composite 3/8 Simpson's rule on $\int_0^1 f(x)dx$ using 7 equally distributed nodes $\{x_i\}_{i=0}^6$, where $x_i = i/6$.

(c) (10 points) Suppose $\max_{0 \le x \le 1} |f^{(4)}(x)| = 1$. Give an upper bound on the error for the composite rule in (b).

5. We approximate the integrand $\int_{-1}^{1} f(x)dx$ by a Gauss quadrature rule $Q[f]$:

$$\int_{-1}^{1} f(x)dx \approx Q[f] = \sum_{i=0}^{n} w_i f(x_i),$$

where the $n+1$ nodes $\{x_i\}_{i=0}^n$ and weights $\{w_i\}_{i=0}^n$ are to be determined.

(a) (5 points) What is the minimum n to guarantee $Q[f]$ is exact for all $f \in \mathbb{P}_9$.

(b) (10 points) Take $n = 2$. Find the nodes $\{x_i\}_{i=0}^2$ and weights $\{w_i\}_{i=0}^2$ of the quadrature rule that maximizes the algebraic accuracy.

List of classical orthogonal polynomials

- Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$: $P_0(x) = 1, P_1(x) = x$, and

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \quad \text{for } n = 1, 2, \cdots .$$

They are orthogonal in $[-1, 1]$ with respect to the standard L^2 inner product:

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}.$$

- Chebyshev polynomials $\{T_n(x)\}_{n=0}^{\infty}$: $T_0(x) = 1, T_1(x) = x$, and

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \quad \text{for } n = 1, 2, \cdots .$$

They are orthogonal in $[-1, 1]$ with respect to L^2_w inner product with weight $w(x) = \frac{1}{\sqrt{1-x^2}}$:

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n \neq 0 \end{cases}.$$

- Laguerre polynomials $\{L_n(x)\}_{n=0}^{\infty}$: $L_0(x) = 1, L_1(x) = 1 - x$, and

$$L_{n+1}(x) = \frac{2n+1-x}{n+1} L_n(x) - \frac{n}{n+1} L_{n-1}(x), \quad \text{for } n = 1, 2, \cdots .$$

They are orthogonal in $[0, \infty)$ with respect to L^2_w inner product with weight $w(x) = e^{-x}$:

$$\int_{-1}^{1} L_m(x) L_n(x) e^{-x} dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}.$$

- Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$: $H_0(x) = 1, H_1(x) = x$, and

$$H_{n+1}(x) = x H_n(x) - n H_{n-1}(x), \quad \text{for } n = 1, 2, \cdots .$$

They are orthogonal in $[-1, 1]$ with respect to L^2_w inner product with weight $w(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$:

$$\int_{-1}^{1} H_m(x) H_n(x) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) dx = \begin{cases} 0 & m \neq n \\ n! & m = n \end{cases}.$$