AMSC/CMSC 460 Computational Methods

Exam 1, Thursday, February 26, 2015 Name: ________________________

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, scratchpaper, etc. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.
   (a) ______ Gauss elimination is the most efficient method to solve a linear system $Ax = b$ where A is a sparse nonsingular square matrix.
   (b) ______ Matlab script `cond(A,1)` provides the condition number of matrix A. It represents how much the absolute error of a vector is amplified by solving the linear system.
   (c) ______ Suppose A is an m-by-n matrix with $m > n$. Then, there must be no solution for $Ax = b$.
   (d) ______ Suppose Q is an n-by-n orthonormal matrix. Then $Q^TQ = \mathbb{I}$ and $QQ^T = \mathbb{I}$. Here, $\mathbb{I}$ is the n-by-n identity matrix.
   (e) ______ Secant method has a superlinear convergence rate for all initials.

2. (20 points) Consider the following Matlab code.

```
A = [.5 4 .5; 1 2 -1;-.2 .8 2.6];
b = [1.5 -1 5]';

c = norm(A,inf)
[L,U,p] = lu(A,'vector')
x = A\b
```

   Find the output `c`, `L`, `U`, `p`, `x` by hand.

3. (a) (12 points) Prove that the two vector norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Namely, prove that for every vector $v \in \mathbb{R}^n$,

$$\|v\|_2 \le \|v\|_1 \le \sqrt{n}\|v\|_2.$$

   Hint: You can apply the following Cauchy-Schwarz inequality on $a = v$ and $b = sgn(v)$ for the right inequality.

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \|a\|_2\|b\|_2, \quad \forall a, b \in \mathbb{R}^n.$$

(b) (8 points) Prove that the two matrix norms $\|\cdot\|_1$ and $\|\cdot\|_2$ induced from the corresponding vector norms are equivalent. Namely, prove that for every n-by-n matrix A,

$$\frac{1}{\sqrt{n}}\|A\|_2 \le \|A\|_1 \le \sqrt{n}\|A\|_2.$$

4. (a) (10 points) Find a QR decomposition of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 8 & -1 \\ 0 & 2 & 3 \\ -2 & 0 & 4 \end{pmatrix}.$$

   (b) (10 points) Use the decomposition in (a) to find a vector $x \in \mathbb{R}^3$ which minimize $\|Ax - b\|_2$ where $b = [1, 1, -1, -1]^T$.

5. The following root finding method is called *relaxation method*. The iterative procedure is given by

$$x_{k+1} = x_k - \lambda f(x_k),$$

where λ is a constant to be determined.

   (a) (5 points) Check the consistency condition: if x_k converges, then the limit must be a root of f.

   (b) (15 points) Suppose f is a smooth function, and $f'(x) \in [1, 2]$. Prove that the scheme converges for $0 < \lambda < 1$. What is the rate of convergence?

   (c) (5 points) If we allow $\lambda = \lambda(x_k)$ that depends on x_k, find an expression of $\lambda(x_k)$ so that the new scheme has a higher rate of convergence. (You should be familiar with the new method.)