Extrapolation in Numerical Integration

Romberg Integration

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Introduction

Extrapolation: the process of estimating beyond the original observation range, the value of a variable on the basis of its relationship with another variable.

(Interpolation is a method of constructing new data points within the range of discrete known points)

Trapezoidal rule is one technique for approximating the integral by approximating the region under the graph of the function as a trapezoid and calculating its area.

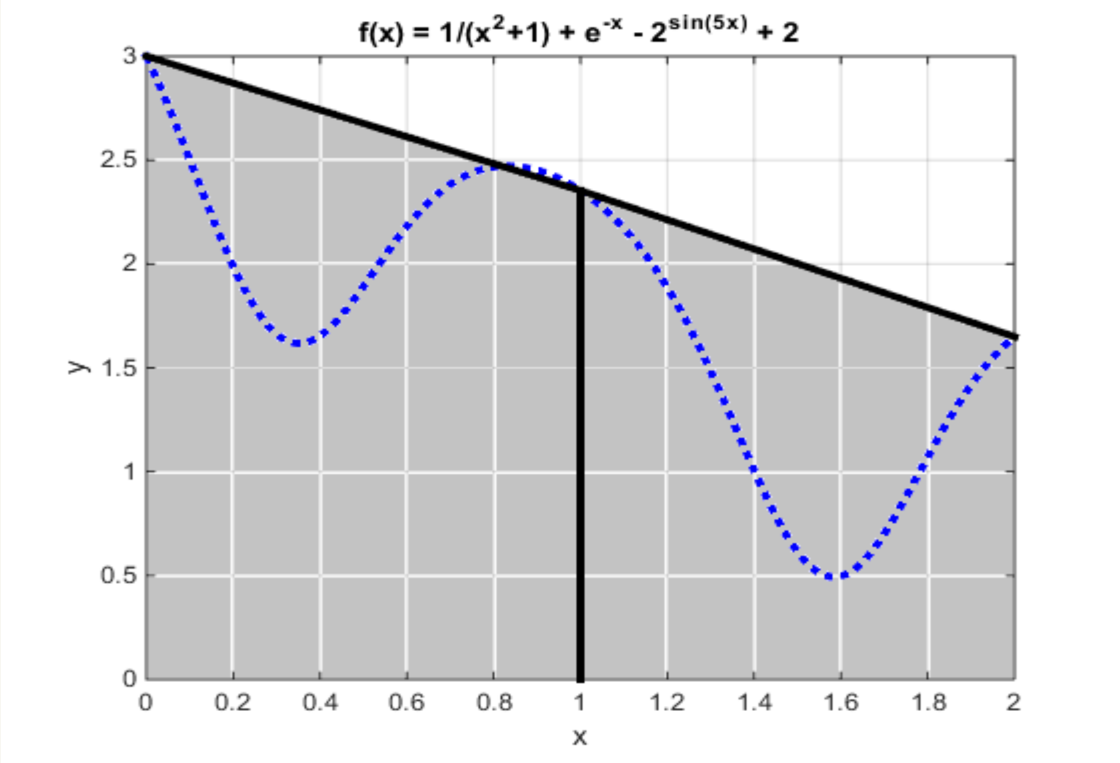
$$\int_a^b f(x)\,dx \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Trapezoidal Rule

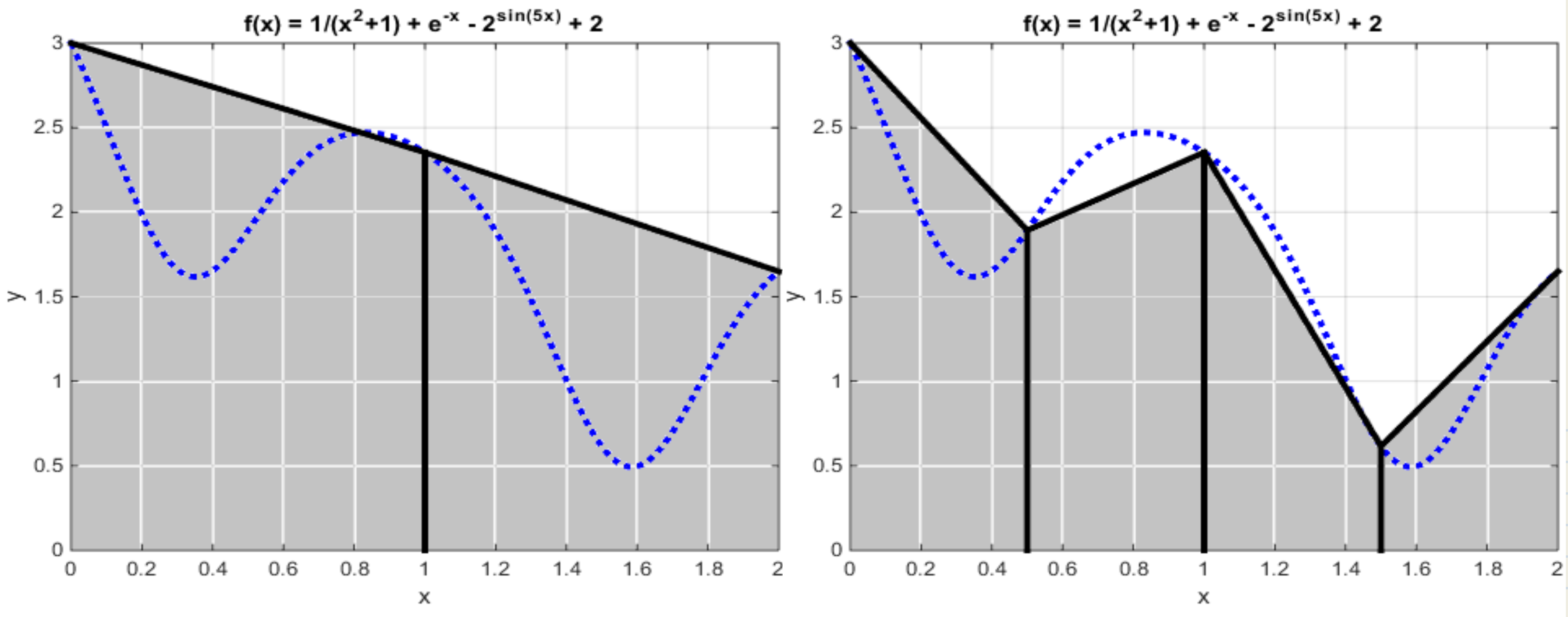
Composite trapezoidal rule divides the range into n equal segments and applies the trapezoidal rule over each segment. The sum of the results obtained for each segment is the approximate value for the integral.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2N}\left(f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + \cdots + 2f(x_N) + f(x_{N+1})\right)$$

Examples



Examples



f(x) = 1/(x^2+1) + e^-x - 2^sin(5x) + 2
y
x
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
f(x) = 1/(x^2+1) + e^-x - 2^sin(5x) + 2
y
x

Trapezoidal Rule (Cont.)

Error of the trapezoidal rule is the difference between the value of the integral and the numerical results.

$$E(h) \approx -\frac{b-a}{12} h_i^2 \bar{f}''$$

Average value of f'' over the entire interval

Error reduces by $O(h^2)$ as the cell size h is decreased with the composite trapezoid rule

The way the trapezoidal rule is derived can be generalized to higher degree polynomial interpolants. Such a quadrature rule is called a Newton-Cotes formula.

Euler-Maclaurin Formula

Euler-Maclaurin summation formula is used to give an estimation of the sum of f(x) through the integral of f(x) with an error term given by an integral => Same idea as in numerical integration

Formula:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) + \sum_{i=2}^{k} \frac{b_i}{i!}(f^{(i-1)}(b) - f^{(i-1)}(a)) - \int_a^b \frac{B_k(\{1-t\})}{k!} f^{(k)}(t)dt$$

Which in case of Trapezoid rule, can be simplified as:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n}[f(x_{i-1}) + f(x_i)] + f(b)\right] + \sum_{r=1}^{k} c_r h^{2r}[f^{(2r-1)}(b) - f^{(2r-1)}(a)] - \left(\frac{h}{2}\right)^{2k} \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} q_{2k}(t) f^{(2k)}(x)\,dx,$$

Trapezoid rule and error

- This formula can explain why Trapezoid rule works very well on periodic functions.

-It also is a different way to analyze error of Trapezoid rule

Romberg's Method

Richardson extrapolation uses multiple subinterval lengths to "extrapolate" better approximations. It jumps to a new, improved approximation using an approximation of the error (E(h) expanded into a series form) .

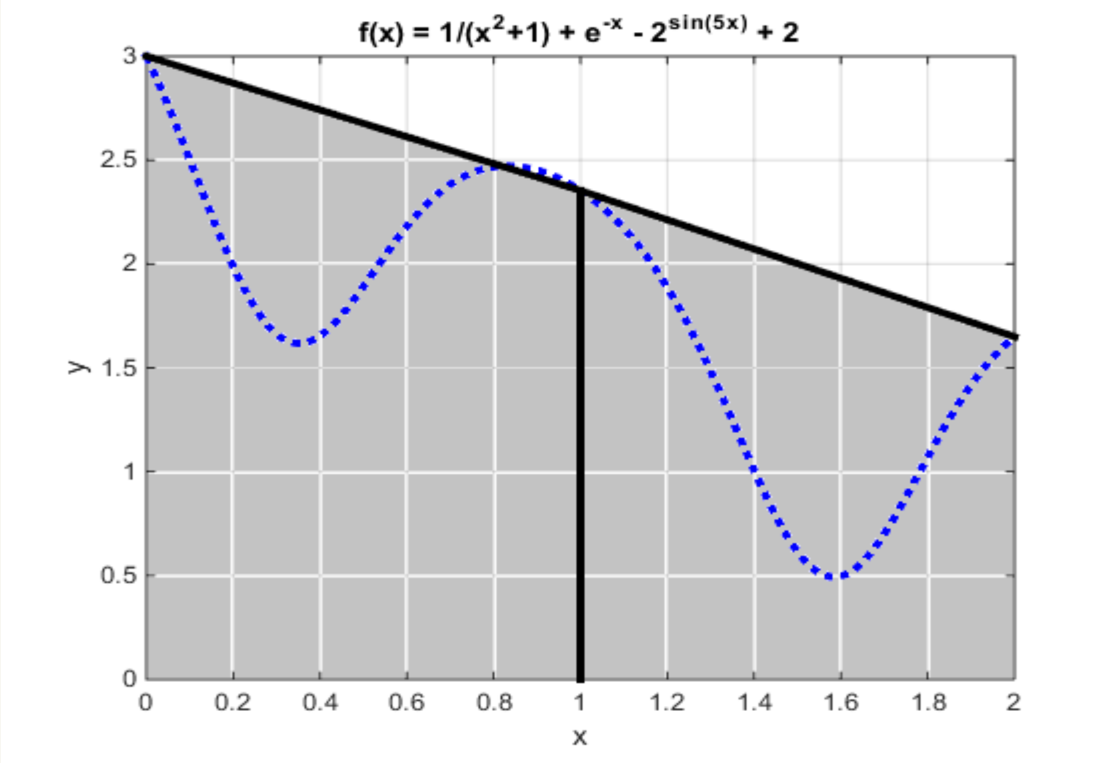
$$E(h) = c_1 h + c_2 h^2 + c_3 h^3 + c_4 h^4 + \cdots$$

Romberg's method is one Newton-Cotes formula which applies Richardson extrapolation repeatedly on the trapezoidal rule to provide a better approximation of the integral by reducing the Error.

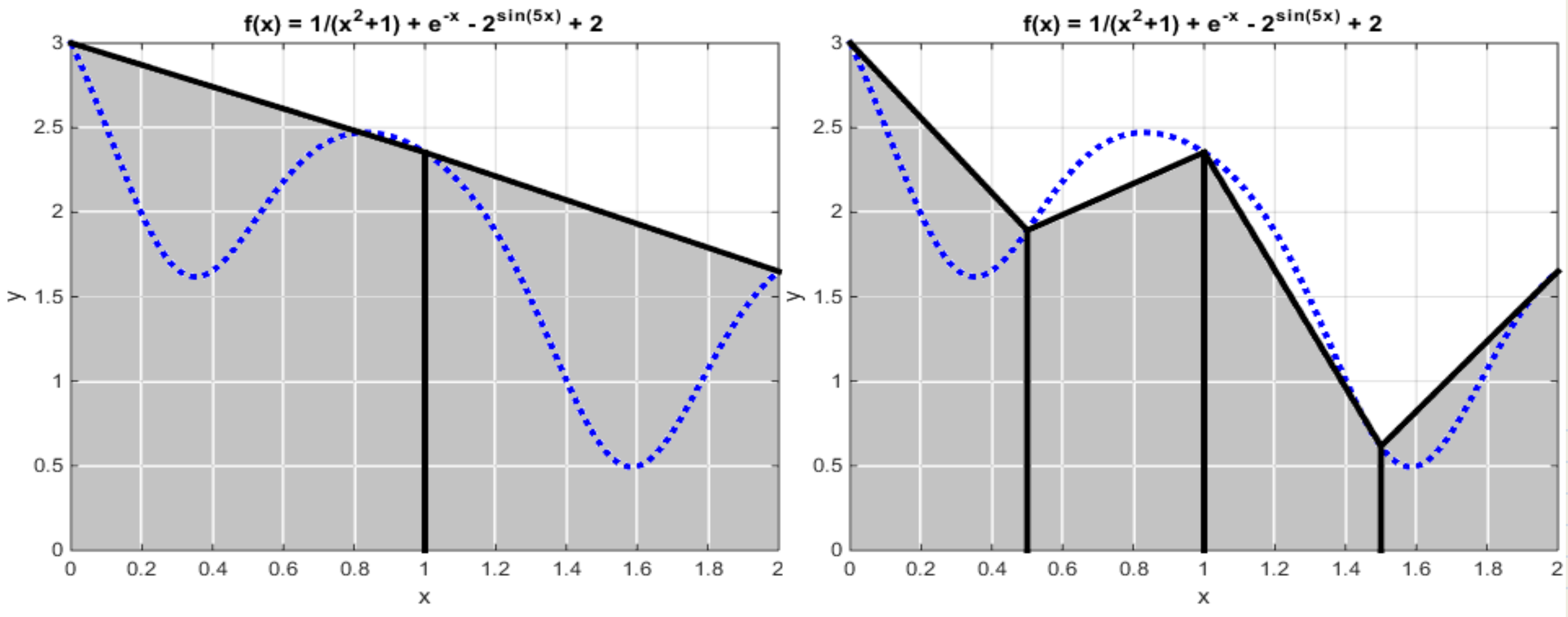
$$I = \int_a^b f(x)\,dx = T(h) + E(h), \qquad \text{with } E(h) = O(h^2)$$

$$E(h) = c_2 h^2 + c_4 h^4 + c_6 h^6 + c_8 h^8 + \cdots$$

Revisiting Trapezoidal Rule



Using Two Estimates



Derivation of Richardson Extrapolation

So first, what is the true value of the actual integral?

It is exactly equal to the following (based on step size h):

$$I = I(h) + E(h)$$

True value — Approximation of integral — Error in approximation

Therefore, with differing step sizes h_i and h_j, such as in the previous figures:

$$I = I(h_i) + E(h_i) = I(h_j) + E(h_j)$$

Derivation (Cont.)

So what would be a good estimate for our error terms?

First, let's expand our E(h) based on what we know from the Trapezoid Rule

$$E(h) \approx -\frac{b-a}{12} h_i^2 \bar{f}''$$

Average value of f'' over the entire interval

Derivation (Cont.)

Taking the ratio of the two error terms $E(h_i)$ and $E(h_j)$ mentioned previously,

$$\frac{E(h_i)}{E(h_j)} \approx \frac{-\frac{b-a}{12} h_i^2 \bar{f}''}{-\frac{b-a}{12} h_j^2 \bar{f}''}$$

Which, after cancelling out common terms, will simplify down to:

$$\frac{E(h_i)}{E(h_j)} \approx \frac{h_i^2}{h_j^2}$$

or

$$E(h_i) \approx \frac{h_i^2}{h_j^2} E(h_j)$$

Derivation (Cont.)

Now that we have separated the $E(h_i)$ error term, we can rearrange the original equation for the true value for the integral as such:

$$E(h_i) - E(h_j) = I(h_j) - I(h_i)$$

By combining the expression for $E(h_i)$ with the above equation, we get:

$$\frac{h_i^2}{h_j^2} E(h_j) - E(h_j) \approx I(h_j) - I(h_i)$$

Which then simply factors out to:

$$E(h_j)(\frac{h_i^2}{h_j^2} - 1) \approx I(h_j) - I(h_i)$$

Derivation (Cont.)

Finally, we've obtained the approximation for the error term $E(h_j)$ in terms of values we already have

$$E(h_j) \approx \frac{I(h_j) - I(h_i)}{\frac{h_i^2}{h_j^2} - 1}$$

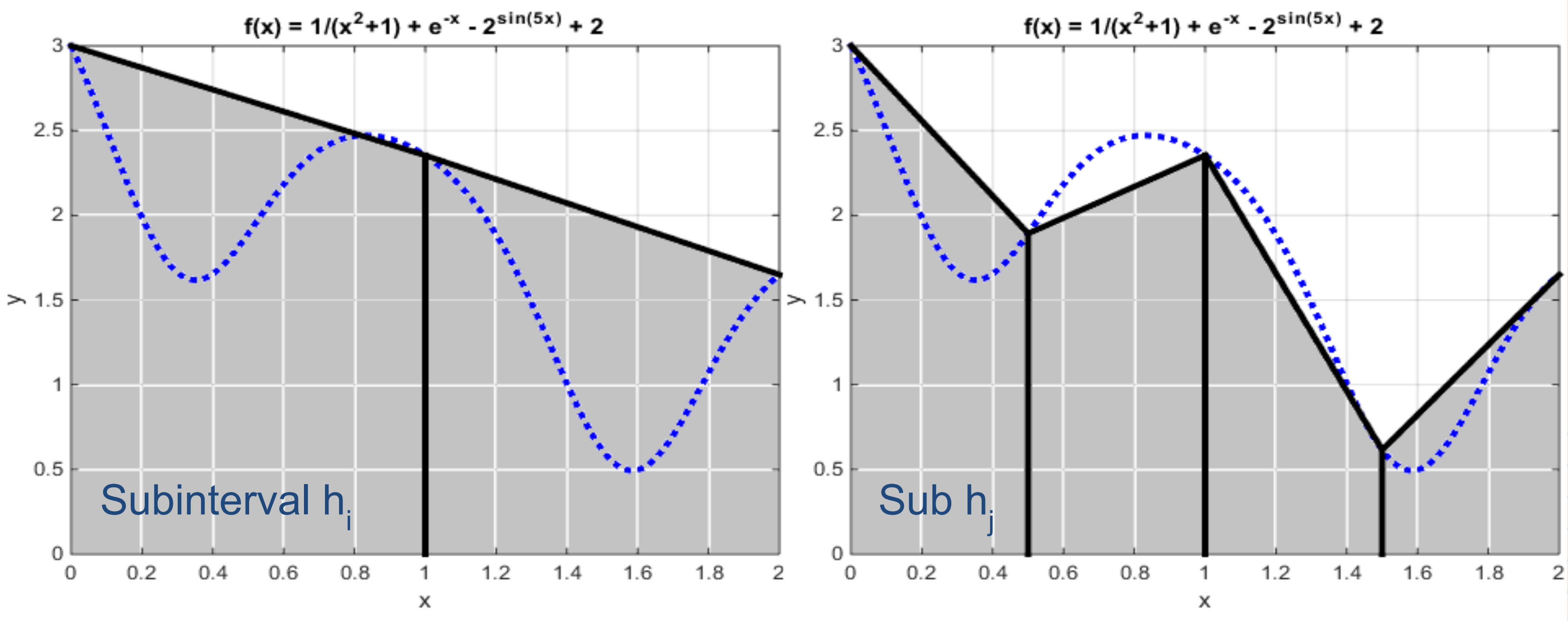
By substituting the above expression into the previous equation,

$$I = I(h_j) + E(h_j)$$

We will now have an improved estimate for the true value of the integral:

$$I \approx I(h_j) + \frac{I(h_j) - I(h_i)}{\frac{h_i^2}{h_j^2} - 1}$$

Going back...



f(x) = 1/(x²+1) + e⁻ˣ - 2^sin(5x) + 2
Subinterval h_i
f(x) = 1/(x²+1) + e⁻ˣ - 2^sin(5x) + 2
Sub h_j
x
y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.5
1.5
2.5
3

Weighted Estimate

With $h_j = h_i/2$, we get the following:

$$I \approx I(h_j) + \frac{I(h_j) - I(h_i)}{\frac{2^2}{1^2} - 1}$$

$$I \approx I(h_j) + \frac{1}{3}I(h_j) - \frac{1}{3}I(h_i)$$

And finally,

$$I \approx \frac{4}{3}I(h_j) - \frac{1}{3}I(h_i)$$

Romberg Integration

When each of the trapezoidal estimates are computed by repeatedly halving the step size (h), we get the general equation for Romberg integration:

$$I_{j,k} \approx \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

Order of Accuracy

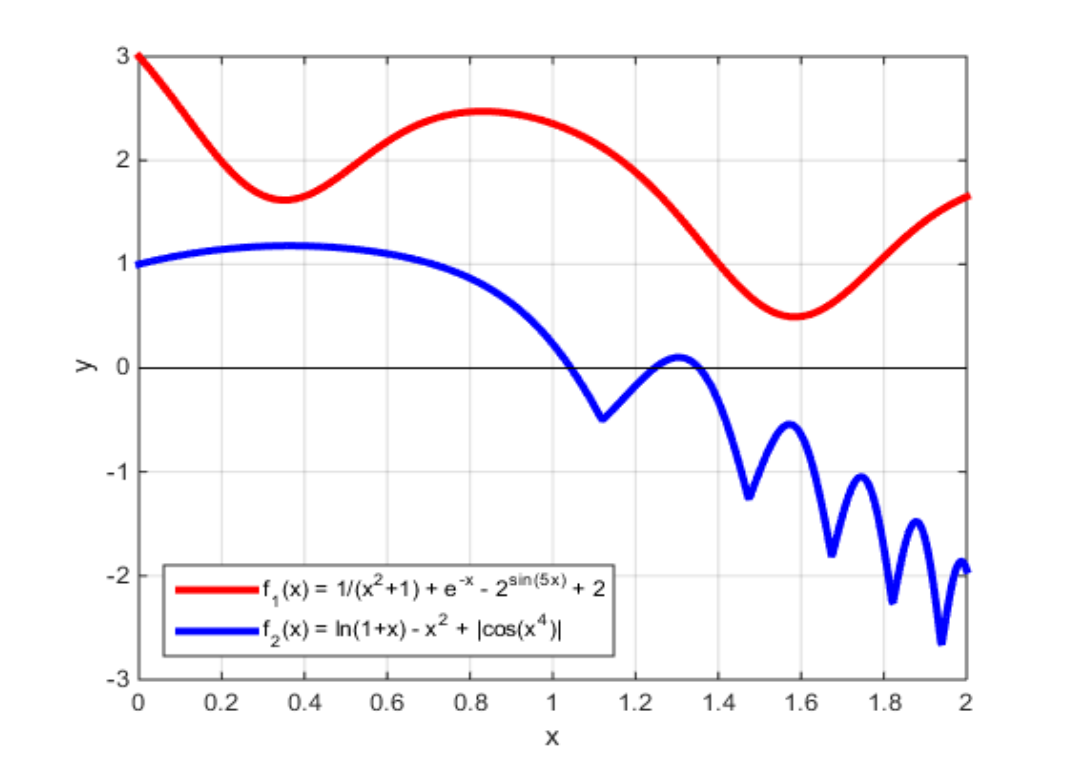
$$\int_0^2 \left[\frac{1}{x^2+1} + e^{-x} - 2^{\sin(5x)} + 2\right] dx \approx 3.4659$$

Step Size (h)	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$
2	4.6781784	3.2309731	3.4820997	3.4655880	3.4658774
1	3.5927745	3.4664043	3.4658460	3.4658763	
0.5	3.4979969	3.4658809	3.4658758		
0.25	3.4739099	3.4658773			
0.125	3.4678846				

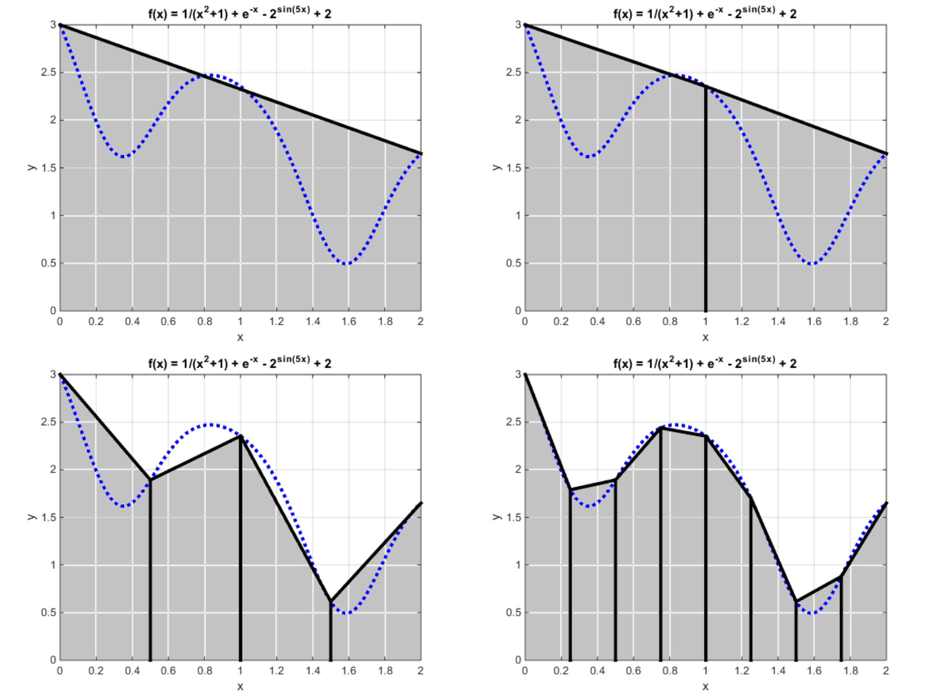
True Value: 3.4658774

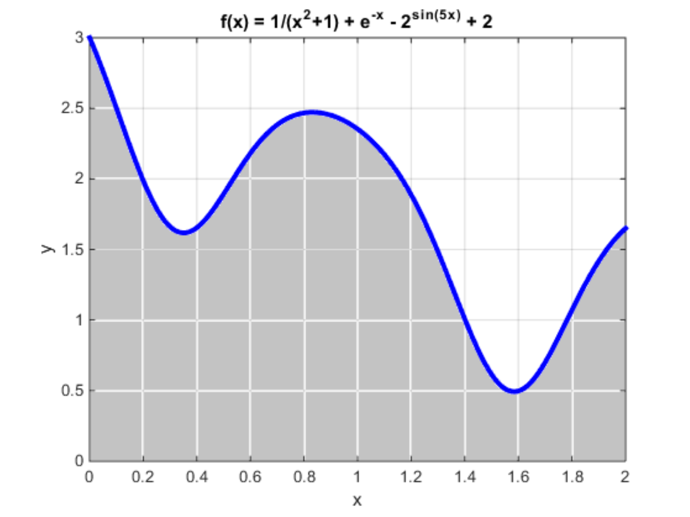
MATLAB Examples

- Implementation of Romberg Integration on two types of functions

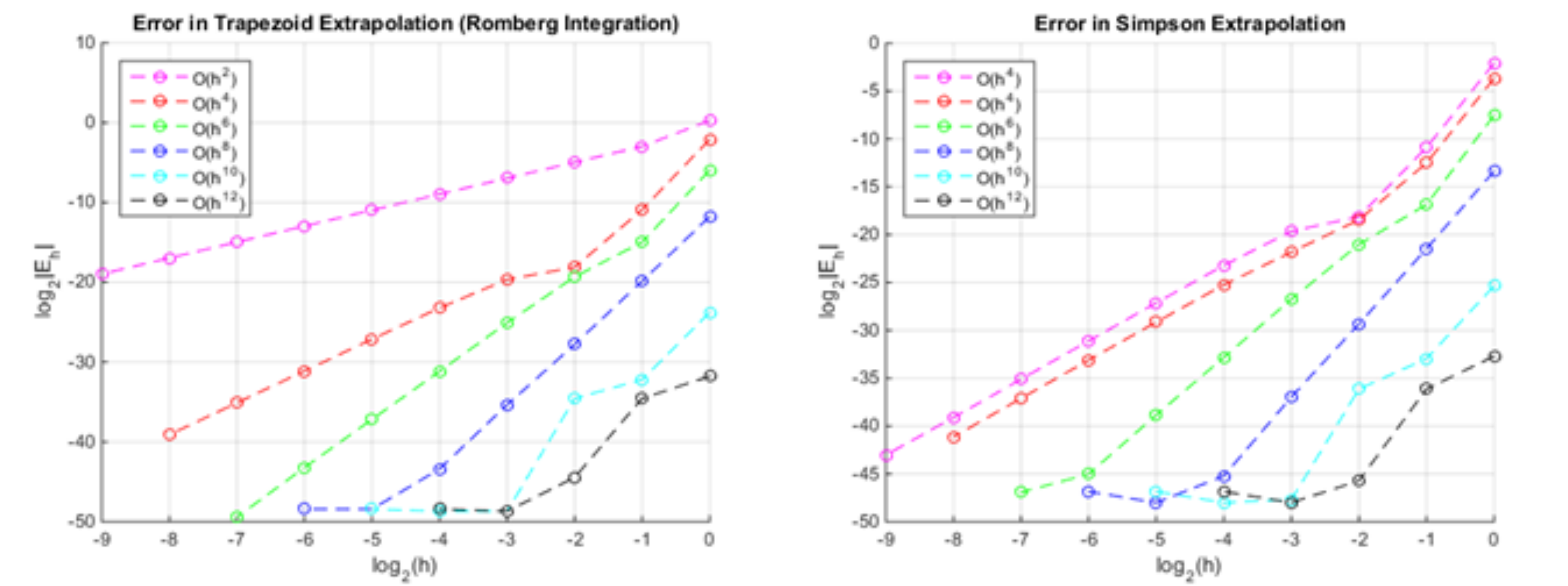


Continuous Function Example



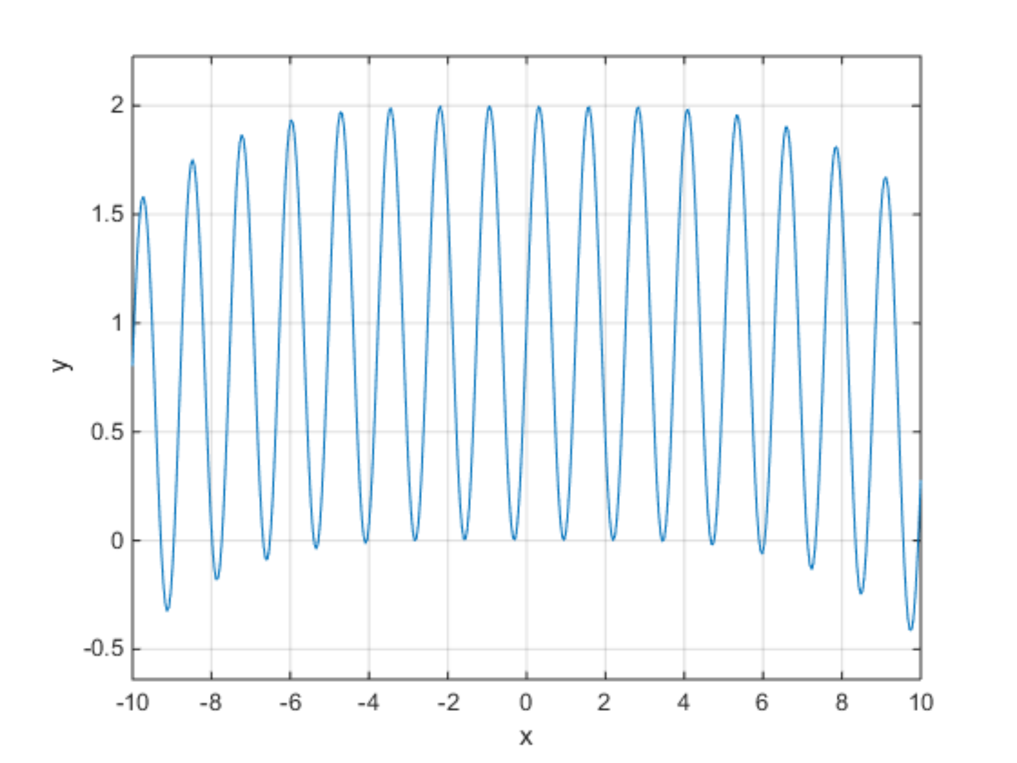


Error in Extrapolations

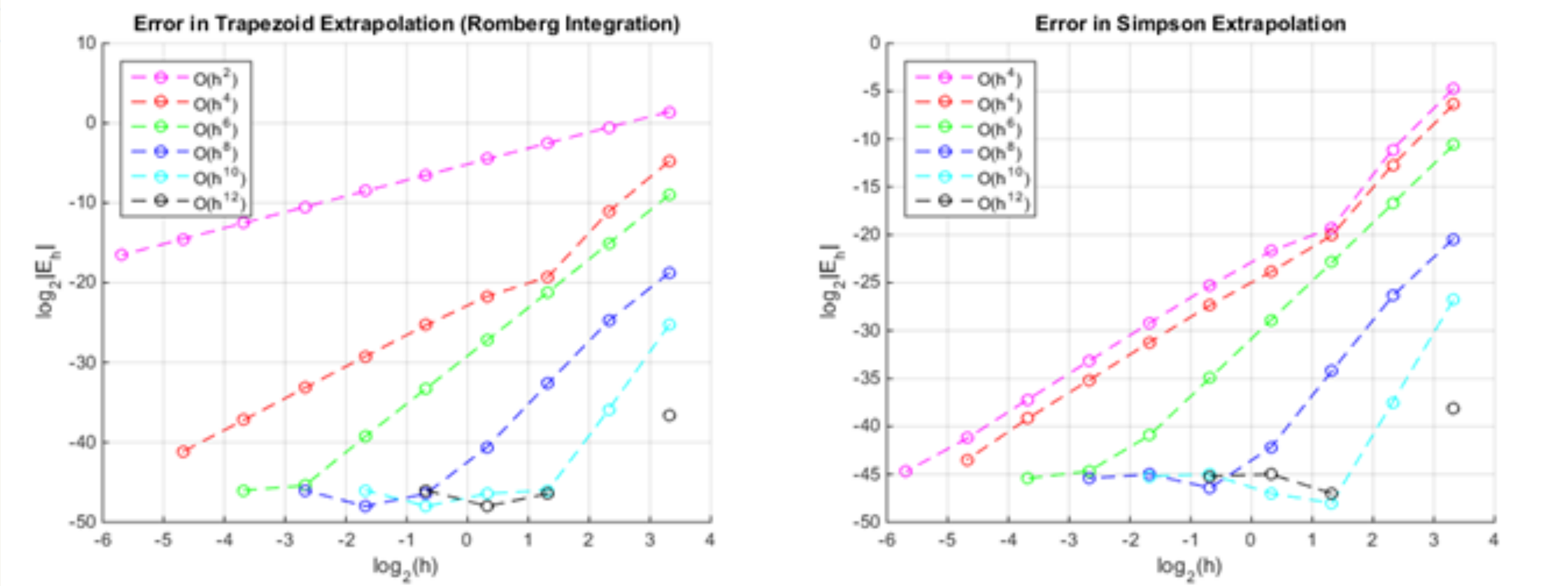


Another Continuous Example

sin(5x) + cos(x^2/100)



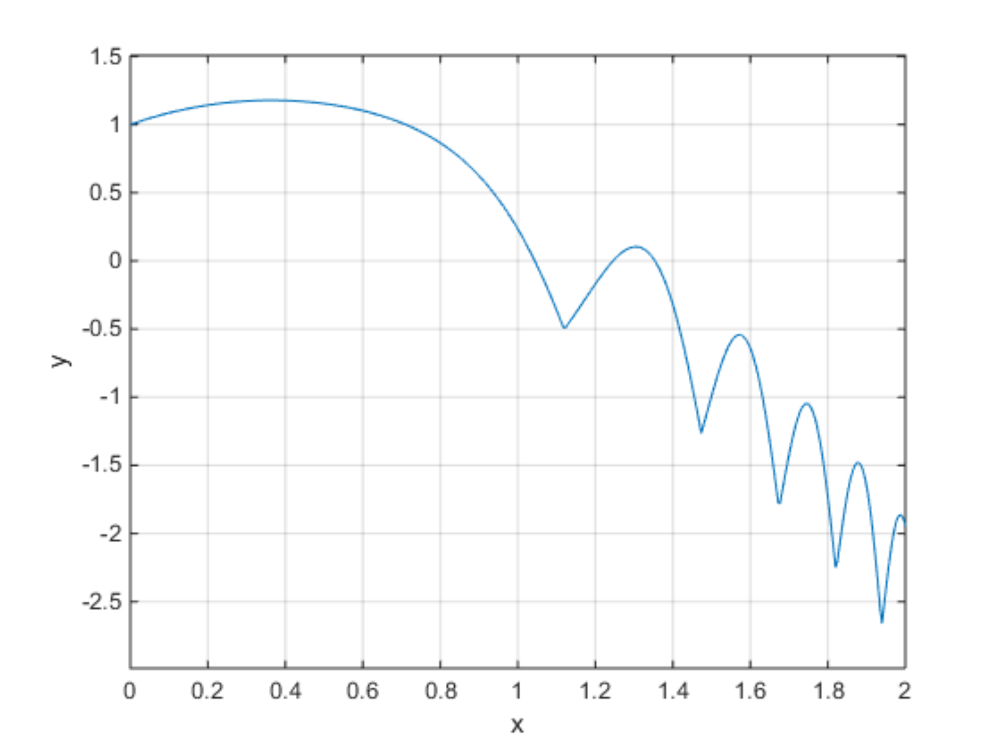
Error in Extrapolations



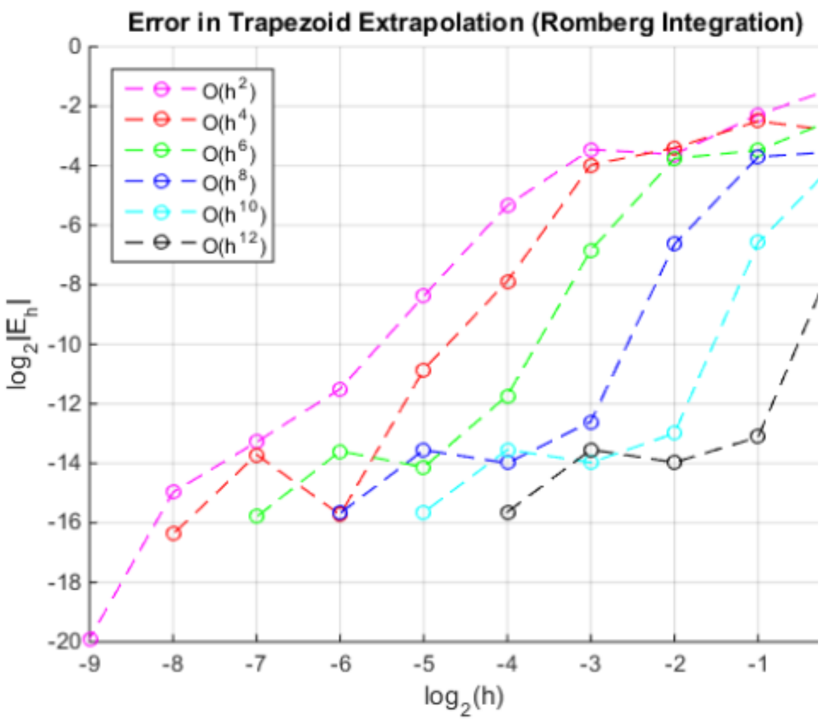
Non-Differentiable functions

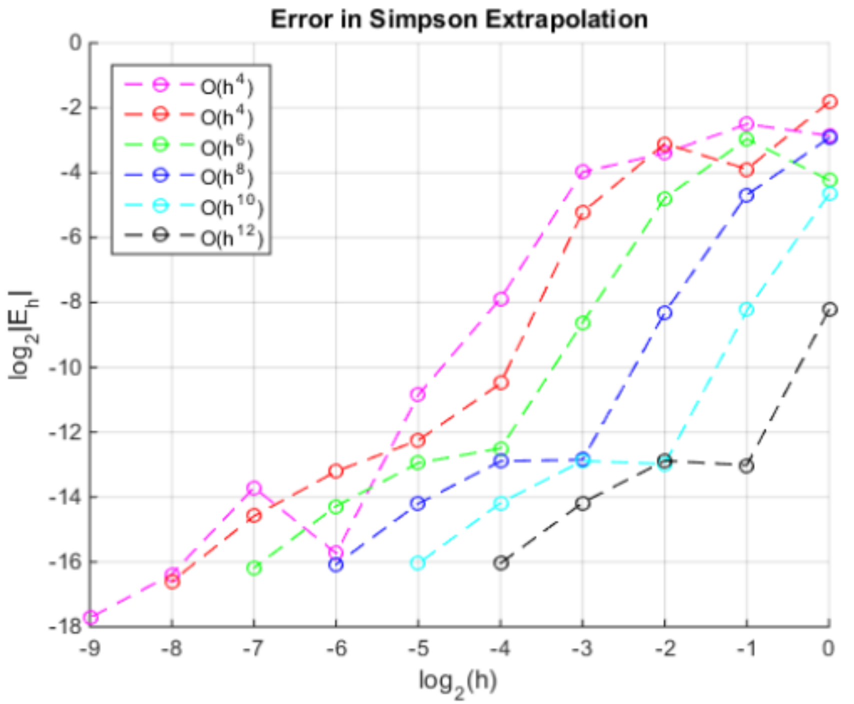
- How much effect does smoothness have on the extrapolations?

ln(1+x)-x^2+abs(cos(x^4))



Error in Extrapolations





Questions?