AMSC/CMSC460 LECTURE NOTES #1

GAUSS ELIMINATION

The purpose of this note is to clarify questions and statements in class, and a step-by-step way to find an LU decomposition by hand.

We start with notations. Let A be an n-by-n non-singular matrix, b is a vector in $\mathbb{R}^n$. The goal is to solve the linear system $Ax = b$, where x is the unique solution of the system.

Define $L_{ij} := \mathbb{I} + l_{ij}E_{ij}$ be the matrix associate to the row operation between i-th and j-th row when performing Gauss elimination. Here, E_{ij} is an n-by-n matrix whose entries are all zeros except the (i,j)-entry is 1. The Gauss elimination procedure (without pivoting) can be described as

$$U = L_{n,(n-1)} \cdots L_{32}L_{31}L_{21}A,$$

where U is an upper triangular matrix. The system is equivalent to

$$Ux = L_{n,(n-1)} \cdots L_{32}L_{31}L_{21}b,$$

and can be solved easily.

The corresponding LU decomposition of A is given by

$$A = LU, \quad \text{where } L = L_{21}^{-1}L_{31}^{-1}L_{32}^{-1} \cdots L_{n,(n-1)}^{-1}.$$

We get U and L_{ij} directly from the Gauss elimination procedure. Here, we derive a way to calculate L by hand.

First, we compute the inverse of L_{ij}.

Proposition 1. *For $i > j$, $L_{ij}^{-1} = \mathbb{I} - l_{ij}E_{ij}$.*

Proof. We check the following identity

$$L_{ij}L_{ij}^{-1} = (\mathbb{I} - l_{ij}E_{ij})(\mathbb{I} - l_{ij}E_{ij}) = \mathbb{I} - l_{ij}^2E_{ij}^2 = \mathbb{I}.$$

Here, we use the fact that $E_{ij}^2 = 0$ if $i \neq j$, which is a direct consequence of proposition 2. □

Example 1. *For $L_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$, its inverse $L_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$.*

Proposition 2. *The matrix E_{ij} satisfies the following identity.*

$$E_{i_1j_1}E_{i_2j_2} = \begin{cases} 0 & \text{if } j_1 \neq i_2 \\ E_{i_1j_2} & \text{if } j_1 = i_2 \end{cases}$$

This in particular implies $L_{i_1j_1}$ and $L_{i_2j_2}$ are commutable if $i_2 \neq j_1$ and $i_1 \neq j_2$. *Note: they do not always commute.*

Next, we compute the product of L_{ij}^{-1}.

Proposition 3. *If $j_1 \neq i_2$, then $L_{i_1j_1}^{-1}L_{i_2j_2}^{-1} = I - l_{i_1j_1}E_{i_1j_1} - l_{i_2j_2}E_{i_2j_2}$.*

Example 2. *If* $L_{21}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $L_{31}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, *then* $L_{21}^{-1}L_{31}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

As $L = L_{21}^{-1}L_{31}^{-1}L_{32}^{-1}\cdots L_{n,(n-1)}^{-1}$, we are always in the case that $j_1 < i_2$. Hence, we can simply assemble each term together to obtain L.

Example 3 (Complete procedure of LU decomposition (without pivoting))**.** *Find the LU decomposition of the following matrix* $A = \begin{pmatrix} 20 & 30 & -10 \\ 6 & 49 & -13 \\ 4 & 26 & 3 \end{pmatrix}$.

Solution. We proceed with Gauss elimination.

$$\begin{matrix} ① \\ ② \\ ③ \end{matrix} \begin{pmatrix} 20 & 30 & -10 \\ 6 & 49 & -13 \\ 4 & 26 & 3 \end{pmatrix} \longrightarrow \begin{matrix} ① \\ ② + (-\frac{6}{20}) \times ① \\ ③ \end{matrix} \begin{pmatrix} 20 & 30 & -10 \\ 0 & 40 & -10 \\ 4 & 26 & 3 \end{pmatrix} \qquad L_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{matrix} ① \\ ② + (-\frac{6}{20}) \times ① \\ ③ + (-\frac{4}{20}) \times ① \end{matrix} \begin{pmatrix} 20 & 30 & -10 \\ 0 & 40 & -10 \\ 0 & 20 & 5 \end{pmatrix} \qquad L_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -.2 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} ① \\ ② \\ ③ \end{matrix} \begin{pmatrix} 20 & 30 & -10 \\ 0 & 40 & -10 \\ 0 & 20 & 5 \end{pmatrix} \longrightarrow \begin{matrix} ① \\ ② \\ ③ + (-\frac{20}{40}) \times ② \end{matrix} \begin{pmatrix} 20 & 30 & -10 \\ 0 & 40 & -10 \\ 0 & 0 & 10 \end{pmatrix} \qquad L_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -.5 & 1 \end{pmatrix}$$

Therefore, we conclude $U = \begin{pmatrix} 20 & 30 & -10 \\ 0 & 40 & -10 \\ 0 & 0 & 10 \end{pmatrix}$, and

$$L = L_{21}^{-1}L_{31}^{-1}L_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ .3 & 1 & 0 \\ .2 & .5 & 1 \end{pmatrix}.$$

□