AMSC/CMSC460 Computational Methods **Fall 2014**

Homework 7, Due on Tuesday, November 4, 2014

1. *(Simpson's rule)* Let f be a $\mathcal{C}^4$ function defined in an interval $[a, b]$. $P_2(x)$ is a polynomial of degree 2 that interpolates f at points $x_0 = a, x_1 = (a + b)/2$ and $x_2 = b$. Simpson's rule is a quadrature rule that approximate $\int_a^b f(x)dx$ by $\int_a^b P_2(x)dx$.

a). Check that the approximated value is in the following form

$$\int_a^b f(x)dx \approx \int_a^b P_2(x)dx = \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right].$$

b). Φ is the cumulative distribution function for standard normal distribution, given as

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{y^2}{2}}dy.$$

Use Simpson's rule to approximate $\Phi(1)$. What is the approximate value? Use Matlab to check the error (take `normcdf(1)` as the "exact" value of $\Phi(1)$).

c). Read the proof of Theorem 7.2 in Suli's book, and use the result to obtain an error bound for your approximation. Verify that the actual error is smaller than this error bound.

d). Let $\{x_i\}_{i=0}^{2m}$ be equally distributed points in $[a, b]$, namely

$$x_i = a + ih, \quad h = \frac{b-1}{2m}.$$

(Note here we redefine x_0, x_1, x_2.) Write a composite Simpson rule. Use Matlab to approximate $\Phi(1)$ by the rule with $m = 2^s$ for $s = 1, 2, \cdots, 8$. Store the error as a sequence (vector) $\{e(s)\}_{s=1}^8$. Display the sequence $\log_2(e(s))$. What do you observe?

Hint: you should observe the difference between the 2 adjacent terms are very close to -4. Can you explain why this happens?

2. *(Quadrature by Hermite interpolation)* Finish Exercise 7.11 in Suli's book in quadrature obtained by Hermite interpolation. What is the corresponding composite rule?

3. *(Gauss quadrature)* The goal of the exercise is to use Gauss quadrature to solve the integral

$$\mathbb{E}(f(X)) = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right)dx,$$

representing the expectation of $f(X)$ where X is a random variable that has a standard normal distribution.

To this end, we introduce *Hermite polynomials* $\{H_n(x)\}_{n=0}^{\infty}$, which are orthogonal with respect to the weight $w(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, namely,

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right)dx = \begin{cases} n! & m = n \\ 0 & m \neq n \end{cases}.$$

The first four Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x.$$

a). We approximate $\mathbb{E}(f(X))$ by Gauss quadrature with 3 nodes, namely

$$\int_{-\infty}^{\infty} f(x) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2).$$

Find the nodes $\{x_i\}_{i=0}^2$ and weights $\{w_i\}_{i=0}^2$ such that the approximation is precise for all $f \in \mathbb{P}_5$.

b). Calculate (by hand) the fourth moment of standard normal distribution $\mathbb{E}(X^4)$ by the quadrature you obtained in a). By fourth moment, we mean $f(x) = x^4$.

Hint: Since $f \in \mathbb{P}_5$, we should expect to get the precise value $\mathbb{E}(X^4) = 3$.

c). Use the same quadrature to approximate $\mathbb{E}(|X|^3)$, where $f(x) = |x|^3$. Do you get the exact value $\mathbb{E}(|X|^3) = \sqrt{\frac{8}{\pi}}$? Briefly explain why you get or not get the precise value.