AMSC/CMSC 460 Computational Methods

Exam 3, Thursday, December 4, 2014

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.
   (a) ______ The formula $y_{n+1} = y_n + hf_{n+1}$ represents an implicit method for solving ODEs.
   (b) ______ Matlab function `ode45` should not be used for stiff systems of differential equations.
   (c) ______ One can design a linear implicit third order scheme for solving ODEs which is A-stable.
   (d) ______ Suppose a multistep scheme has a truncation error of order s, and it is stable. Then, any algorithm with this scheme has convergence with order s.
   (e) ______ The projection of a vector u onto v can be calculated by $\frac{\langle u, v\rangle}{\|u\|^2}v$, where $\langle\cdot,\cdot\rangle$ denotes the inner product.

2. (20 points) The following Matlab code solves the ODE $y' = f(x, y)$.

```
for i = 1:N+1
    k1 = f(x(i),y(i));
    k2 = f(x(i)+3/4*h,y(i)+3/4*h*k1);
    y(i+1) = y(i)+h/3*(k1+2*k2);
end
```

   Write down the scheme, and find the order of accuracy.

3. (a) (10 points) A trapezoid rule method is given as

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})].$$

   Write down the trancation error, and prove that the method has second order accuracy.
   Hint: You are free to use the error estimate for trapezoid rule:

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}(f(a) + f(b))\right| \le \frac{(b-a)^3}{12}\max_{\xi\in[a,b]}|f''(\xi)|.$$

(b) (5 points) Consider the initial value problem

$$\begin{cases} y' = -\lambda y \\ y(0) = y_0. \end{cases}$$

Write the trapezoid rule explicitly.

(c) (10 points) For $\lambda > 0$, the exact solution of the initial value problem is $y(x) = y_0 e^{-\lambda x}$, which decays as x becomes larger. Prove that the method is A-stable, namely, for any stepsize $h > 0$, $|y_{n+1}| < |y_n|$.

(d) (5 points) Does the method converge? What is the rate of convergence? (Just state the result. No need to prove.)

4. Consider the following linear system:

$$\begin{aligned} \alpha - \beta &= 1 \\ -\alpha + 2\beta &= 2 \\ 2\alpha + \beta &= 1 \end{aligned}$$

(a) (5 points) Write the system in matrix form $Ax = b$. Does this system have a solution?

(b) (10 points) Find x that minimizes the error in L^2, namely, $\min_x \|Ax - b\|_2^2$.

5. (20 points) Find a QR docomposition of the following matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 3 \\ 1 & 1 & 5 \end{pmatrix}.$$