AMSC/CMSC 460 Computational Methods

Exam 2, Thursday, November 6, 2014 **Solution**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.

   (a) ______ There exists a unique polynomial of degree up to 5 that interpolates a function at 6 distinct nodes.

   > **Solution:** True. Polynomial of degree 5 has 6 degree of freedom.

   (b) ______ A Hermite cubic spline is a $\mathcal{C}^2$ function.

   > **Solution:** False. It is not twice differentiable at the knots.

   (c) ______ Suppose $s(x)$ is a piecewise linear polynomial with respect to knots $\{x_i\}_{i=0}^m$, which minimizes $\|f-s\|_2$. Then, s interpolates f at the knots.

   > **Solution:** False. The L^2 minimizer does not necessarily interpolate f.

   (d) ______ Matlab script `sparse([1,2],[3,1],[1,2])` generates a sparse matrix $\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$.

   > **Solution:** True.

   (e) ______ Gaussian integration uses equally spaced points.

   > **Solution:** False.

2. $p_2(x)$ is a quadratic polynomial which interpolates $f(x) = (1+|x|)^{-1/2}$ at three nodes $x = -3, 0, 3$.

   (a) (10 points) Express $p_2(x)$ in the form of $ax^2 + bx + c$.

   > **Solution:** p_2 has the property $p_2(-3) = p_2(3) = 1/2$ and $p_2(0) = 1$. Therefore, $p_2(x) = -\frac{x^2}{18} + 1$.

Remark: The obtain p_2, one can use Lagrange polynomial, Newton's representation, or Vandermonde matrix.

(b) (10 points) Obtain an error bound uniformly in $[-3,3]$. Namely, find an upper bound of $\|f-p_2\|_{L^\infty([-3,3])}$.

Solution: The point-wise error formula reads

$$f(x)-p_2(x)=\frac{f'''(\xi)}{6}\pi(x), \quad \pi(x)=(x+3)x(x-3).$$

For uniform estimate, we can only consider $x\in[0,3]$ by symmetry.

$$\|f-p_2\|_{L^\infty([-3,3])}=\|f-p_2\|_{L^\infty([0,3])}\le\frac{\max_{\xi\in[0,3]}|f'''(\xi)|}{6}\max_{x\in[0,3]}|\pi(x)|.$$

In this case, $f'''(\xi)=-\dfrac{15}{8}(1+\xi)^{-1/2} \quad\Rightarrow\quad \max\limits_{\xi\in[0,3]}|f'''(\xi)|\le\dfrac{15}{8}.$

For $\max_{x\in[0,3]}|\pi(x)|$, as $\pi'(x)=3x^2-9$, the stationary points of $\pi(x)$ for $x\in[0,3]$ is $x=\sqrt{3}$. Compare with the endpoints: $\pi(\sqrt{3})=-6\sqrt{3},\pi(0)=\pi(3)=0$. Therefore, $\max_{x\in[0,3]}|\pi(x)|=6\sqrt{3}$.

We conclude that $\|f-p_2\|_{L^\infty([-3,3])}=\frac{15}{8}\cdot\frac{1}{6}\cdot6\sqrt{3}=\frac{15\sqrt{3}}{8}$.

3. (20 points) Find the cubic polynomial $p_3(x)$ which minimizes $\|f-p_3\|_{L^2([-1,1])}$, for function $f(x)=x^4$.

Hint: You might make use of Legendre polynomials $\{P_n(x)\}_{n=0}^\infty$, which are orthogonal with respect to L^2 norm.

$$\int_{-1}^{1}P_m(x)P_n(x)dx=\begin{cases}0 & m\neq n\\ \frac{2}{2m+1} & m=n\end{cases}.$$

The first 4 Legendre polynomials is given as below:

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{2}(3x^2-1),\quad P_3(x)=\frac{1}{2}(5x^3-3x).$$

To save you some computational load, $\displaystyle\int_{-1}^{1}x^4P_2(x)dx=\frac{8}{35}$ *and* $\displaystyle\int_{-1}^{1}x^4P_3(x)dx=0.$

Solution: Write $p_3(x)=\sum_{i=0}^3\alpha_iP_i(x)$. The coefficients satisfy the linear system

$$\begin{pmatrix}\langle P_0,P_0\rangle & \cdots & \langle P_0,P_3\rangle\\ \vdots & \ddots & \vdots\\ \langle P_3,P_0\rangle & \cdots & \langle P_3,P_3\rangle\end{pmatrix}\begin{pmatrix}\alpha_0\\ \ddots\\ \alpha_3\end{pmatrix}=\begin{pmatrix}\langle P_0,f\rangle\\ \vdots\\ \langle P_3,f\rangle\end{pmatrix}.$$

Make use of the orthogonality of Legendre polynomial, we get

$$\begin{pmatrix} 2 & & & \\ & 2/3 & & \\ & & 2/5 & \\ & & & 2/7 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 0 \\ 8/35 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 0 \\ 4/7 \\ 0 \end{pmatrix}.$$

Therefore, $p_3(x) = \frac{1}{5} + \frac{4}{7} \cdot \frac{1}{2}(3x^2 - 1) = \frac{6}{7}x^2 - \frac{3}{35}$.

4. (a) (5 points) Write a composite trapezoid rule on $\int_0^1 f(x)dx$ using 6 equally distributed nodes $\{x_i\}_{i=0}^5$, where $x_i = i/5$.

Solution:

$$\int_0^1 f(x)dx \approx \frac{1}{10}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)].$$

(b) (10 points) Suppose $\max_{0\le x\le 1} |f''(x)| = 1$. Give an upper bound on the error. *Hint: You can use the following error estimate on trapezoid rule without a proof.*

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] \right| \le \frac{(b-a)^3}{12} \max_{x\in[a,b]} |f''(x)|.$$

Solution: Denote E the error in (a).

$$\begin{aligned} |E| &= \left| \sum_{i=1}^{5} \left[\int_{x_{i-1}}^{x_i} f(x)dx - \frac{x_i - x_{i-1}}{2}(f(x_{i-1}) + f(x_i)) \right] \right| \\ &\le \sum_{i=1}^{5} \frac{(x_i - x_{i-1})^3}{12} \max_{x\in[x_{i-1},x_i]} |f''(x)| \\ &\le 5 \cdot \left(\frac{1}{5}\right)^3 \frac{1}{12} \cdot 1 = \frac{1}{300}. \end{aligned}$$

5. Chebyshev polynomials $\{T_n\}_{n=0}^\infty$ are orthogonal with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$. More precisely, they satisfy the following identities.

$$\int_{-1}^1 T_m(x)T_n(x)\frac{1}{\sqrt{1-x^2}}dx = \begin{cases} 0 & m \ne n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n \ne 0 \end{cases}.$$

Chebyshev polynomials can be defined recursively by

$$T_0(x) = 1, \quad T_1(x) = x, \quad \text{and} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \text{ for } n \geq 1.$$

(a) (5 points) Find the first five Chebyshev polynomials $\{T_n\}_{n=0}^4$.

Solution: $T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, T_4(x) = 8x^4 - 8x^2 + 1$.

(b) (15 points) Set up a Gauss quadrature for the following numerical integration.

$$\int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}} dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2).$$

Find the nodes $\{x_i\}_{i=0}^2$ and weights $\{w_i\}_{i=0}^2$ such that the approximation is precise for all $f \in \mathbb{P}_5$.

Solution: The knots are roots for T_3. Hence, $x_0 = -\sqrt{3}/2, x_1 = 0, x_2 = \sqrt{3}/2$. For the weights, we solve the following linear system

$$\begin{pmatrix} T_0(x_0) & T_0(x_1) & T_0(x_2) \\ T_1(x_0) & T_1(x_1) & T_1(x_2) \\ T_2(x_0) & T_2(x_1) & T_2(x_2) \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \langle T_0, T_0 \rangle \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \pi/3 \\ \pi/3 \\ \pi/3 \end{pmatrix}.$$

Therefore, the Gauss quadrature is given as

$$\int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}} dx \approx \frac{\pi}{3} f\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3} f(0) + \frac{\pi}{3} f\left(\frac{\sqrt{3}}{2}\right).$$

(c) (10 points) Use the quadrature rule to find

- the exact value of the integral $\int_{-1}^{1} \frac{x^5 - 1}{\sqrt{1-x^2}} dx$,
- an approximate value of the integral $\int_{-1}^{1} \sqrt{\frac{4x^2+1}{1-x^2}} dx$.

Solution: For the first integral, $f(x) = x^5 - 1 \in \mathbb{P}_5$. Plug in the formula, we get the exact value $-\pi$.
For the second integral, $f(x) = \sqrt{4x^2+1}$. Plug in the formula, we get the approximate value $5\pi/3$.