CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Homework 8, Due on Tuesday, November 14, 2017

1. *(Schrödinger equation)* The initial value problem of linear Schrödinger equation is given as

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

a). Consider plane wave ansatz $u(x,t) = e^{i(y \cdot x - \sigma t)}$. Find the speed of propogation $\sigma/|y|$, and show that the equation is *dispersive*.

b). Write the equation for $\hat{u}$, where Fourier transform is taken in space only. Solve $\hat{u}$.

c). Prove that the L^2-energy is preserved in time, namely $\|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}$.

Hint: Due to Plancherel's theorem, one just need to check $\|\hat{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)}$ preserves in time.

2.(*) *(Galilean invariance of Laplace equation)* Let f be a function in $\mathbb{R}^n$.

a). Prove that Laplace equation is invariant under *translation*: if f is harmonic, namely $\Delta f(x) = 0$, then $f(\cdot + v)$ is harmonic as well, namely $\Delta f(x + v) = 0$, for all $v \in \mathbb{R}^n$.

b). Prove that Laplace operator is invariant under *rotation*: $\Delta f(x) = \Delta f(Ox)$, for all orthonormal $n \times n$ matrix O. It implies that Laplace equation is invariant under rotation.

Remark: This problem is required for MATH513/CAAM523 students only.

3.(**) *(Gauss-Green formula)* Let $u, v \in C^2(\bar{\Omega})$. Prove that

$$\int_\Omega (u\Delta v - v\Delta u)dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}} v \right) dS(x).$$

Remark: This problem is optional. No need to write the proof if you know how. Read page 711-712 in Evans book if you run into difficulties.

4. *(Energy method)* Let Ω be a bounded open set in $\mathbb{R}^n$. Use energy method to prove the following uniqueness arguments.

a). If u and v are two solutions of Poisson equation with Dirichlet boundary condition,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then $u \equiv v$.

b). If u and v are two solutions of Poisson equation with Neumann boundary condition,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega, \end{cases}$$

then $u(x) \equiv v(x) + c$, where c is a constant.