CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Homework 7, Due on Tuesday, November 7, 2017

1. *(Lebesgue spaces)*

a). Find a function f_1 which lies in $L^1(\mathbb{R})$ but not $L^\infty(\mathbb{R})$.

b). Find a function f_2 which lies in $L^\infty(\mathbb{R})$ but not $L^1(\mathbb{R})$.

c). Prove that if $f \in L^1 \cap L^\infty(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$.

2. *(Solve wave equation using Fourier transform)* Consider the initial value problem

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \\ u(x, t = 0) = g, \quad u_t(x, t = 0) = h. \end{cases}$$

Let v be the Fourier transform of u in x, namely

$$v(y, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x, t) e^{-ix \cdot y} dx.$$

Find the initial value problem that v solves. Solve for v explicitly. *Hint: For each given y, $v(y, \cdot)$ solves a second order linear ODE.*

Remark: One can then find the solution $u = \mathcal{F}^{-1}[v]$. In general, it is not easy to find the explicit expression of u.

3. *(Derive d'Alembert's formulation using Fourier transform)* Consider Fourier transform in $\mathbb{R}$ defined by $\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy} dx$.

a). Prove that if f is even, then $\hat{f}$ is real.

b). Let $f(x) = 1_{[-a,a]}(x)$ be the indicator function of interval $[-a, a]$. Calculate $\hat{f}$ explicitly.

c). Consider the following initial value problem of wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = 0, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Use the result in problem 2 and write down $\hat{u}$. Use the properties of Fourier transoform to solve u.

d). Apply Stokes' rule and derive d'Alembert's formula. (See problem 18 in section 2.5 of Evans book for the description of the Stokes' rule. We have used it in the previous homework.)