CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Homework 5, Due on Tuesday, October 24, 2017

1. *(Fundamental solution for 1D wave equation)* Consider the following initial value problem of 1D wave equation

$$\begin{cases} \Phi_{tt} - c^2\Phi_{xx} = 0 \\ \Phi(x, t = 0) = 0, \quad \Phi_t(x, t = 0) = \delta(x) \end{cases},$$

where δ is the Dirac delta distribution.

a). Write the weak formulation of the initial value problem.

b). Verify that the following Φ is a weak solution of the problem:

$$\Phi(x, t) = \begin{cases} \frac{1}{2c} & |x| < ct, t > 0 \\ 0 & |x| > ct, t > 0 \\ 0 & t < 0 \end{cases}$$

c). Show that $u(x, t) = \int_{\mathbb{R}} \Phi_t(x-y, s)g(y)dy + \int_{\mathbb{R}} \Phi(x-y, t)h(y)dy$ solves the following problem

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 \\ u(x, t = 0) = g(x), \quad u_t(x, t = 0) = h(x) \end{cases}.$$

Use this to derive d'Alembert's formula.

2. *(1D non-homogeneous wave equation)* Solve the following initial value problem

$$\begin{cases} u_{tt} - c^2u_{xx} = f(x, t) \\ u(x, t = 0) = 0, \quad u_t(x, t = 0) = 0 \end{cases}.$$

Remark: You can solve it by fractorize the 1D wave operator. Meanwhile, you are encouraged to think about how is the solution related to the fundamental solution. Can you express the solution in terms of Φ and f just like 1c) ?

3. *(Energy estimate and uniqueness)* Consider the following equation characterizing vibrating strings with air resistance

$$u_{tt} - c^2u_{xx} + ru_t = 0, \quad u(x, t = 0) = g(x), \quad u_t(x, t = 0) = h(x).$$

where c is the wave speed, and $r > 0$ is the resistance coefficient.

a). Define *energy* $E(t) = \int_{\mathbb{R}} \left(u_t^2(x, t) + c^2u_x^2(x, t)\right) dx$. Show that $E(t)$ decays in time.

b). Prove that the classical solution of the initial value problem (if exists,) is unique.