CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Homework 4, Due on Tuesday, October 17, 2017

(The problems marked with (**) are optional for everybody, and do not have to be sumitted. Solving them would require additional knowledge which is not covered in class.)

1. *(Noncompatibility of weak solutions under nonlinear transformations)*

a). Find the entropy solution of the Riemann problem

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad u(x,t=0) = \begin{cases} 1 & \text{if } x<0 \\ 0 & \text{if } x>0 \end{cases}.$$

b). Let $v = u^2$. Verify that if u is differentiable, then v satisfies

$$v_t + \left(\frac{2}{3}v^{3/2}\right)_x = 0, \qquad v(x,t=0) = \begin{cases} 1 & \text{if } x<0 \\ 0 & \text{if } x>0 \end{cases}.$$

c). Find the entropy solution of the Riemann problem on v. Does the solution matches u^2?

Note: From this example, one can see the shock speed might change if we perform a nonlinear transformation of the equation. The reason is that chain rule does not work well with weak derivative. Think about $(H^2)_x \neq 2HH_x$ (in the weak sense), where H is the Heavyside function.

2. *(System of 1D conservation law)* Solve the following initial value problem

$$\begin{cases} u_t + (u+2v)_x = 0 \\ v_t + (2u+v)_x = 0 \end{cases} \qquad \text{with} \qquad \begin{cases} u(x,t=0) = \sin x \\ v(x,t=0) = \cos(2x) \end{cases}.$$

3. *(Isothermal Euler equations)* Consider 1D compressible Euler equations with isothermal pressure

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + C\rho)_x = 0 \end{cases}$$

where (ρ, u) represent density and velocity respectively, and $C > 0$ is a constant.

a). Write the system in the form $U_t + F(U)_x = 0$, where $U = (\rho, u)$.

b). Calculate the matrix $\nabla F(U)$ and find its eigenvalues. Show that the system is strictly hyperbolic if $\rho > 0$.

c). Compute the corresponding Riemann invariants $\mathbf{w} = (w_1, w_2)$ of the system.

d). (**)Find the condition that entropy has to satisfy. Can you find an entropy-pair?

e). (**) Describe how you can solve a Riemann problem.