CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Final Exam Solution

1. (20 points) Consider the following initial value problem

$$\begin{cases} (u_{x_1})^2 + \frac{1}{2}(u_{x_2})^2 = \beta + x_1^2, & \text{in } \mathbb{R}^2, \\ u = \frac{x_1^2}{2}. & \text{on } \mathbb{R} \times \{x_2 = 0\}, \end{cases}$$

with $u_{x_2}(x_1, 0) > 0$ for all $x_1 \in \mathbb{R}$; β is a positive constant ($\beta > 0$).

(a) Find the *explicit* solution $u(x_1, x_2)$ of this problem for any given $\beta > 0$.

(b) For what value(s) of β is $u \equiv 0$ on the parabola $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1^2\}$?

Solution: (a). The equation has the form $F(x_1, x_2, z, p_1, p_2) = p_1^2 + \frac{1}{2}p_2^2 - \beta - x_1^2 = 0$. The system of characteristic paths starting from $(\alpha, 0)$ reads

$$\begin{cases} \dot{x}_1 = 2p_1 \\ \dot{x}_2 = p_2 \\ \dot{z} = 2p_1^2 + p_2^2 \\ \dot{p}_1 = 2x_1 \\ \dot{p}_2 = 0 \end{cases} \qquad \text{subject to initial condition} \qquad \begin{cases} x_1(0) = \alpha \\ x_2(0) = 0 \\ z(0) = \frac{\alpha^2}{2} \\ p_1(0) = \alpha \\ p_2(0) = \sqrt{2\beta} \end{cases}$$

where $p_1(0)$ is computed from initial data, and $p_2(0)$ is computed from the equation.

To solve the ODE system, we first observe $p_2(s) = \sqrt{2\beta}$, and then $x_2(s) = \sqrt{2\beta}s$. The coupled dynamics (x_1, p_1) yields $x_1(s) = \alpha e^{2s}$ and $p_1(s) = \alpha e^{2s}$. Plug in everything to the z equation and get $\dot{z} = 2\alpha^2 e^{4s} + 2\beta$, and therefore $z(s) = \frac{\alpha^2}{2}e^{4s} + 2\beta s$. The solution along the characteristic path starting from $(\alpha, 0)$ reads

$$u(x_1(s), x_2(s)) = \frac{\alpha^2}{2}e^{4s} + 2\beta s.$$

Now, we invert the map $(\alpha, s) \to (x_1, x_2)$, and get $s = \frac{x_2}{\sqrt{2\beta}}$, $\alpha = x_1 e^{-2x_2/\sqrt{2\beta}}$. Therefore, the solution of the initial value problem is

$$\boxed{u(x_1, x_2) = \frac{x_1^2}{2} + \sqrt{2\beta}x_2.}$$

(b). On the porabola Γ, $u(x_1, -x_1^2) = \frac{x_1^2}{2} - \sqrt{2\beta}x_1^2$. Clearly, it is equal to zero when $\frac{1}{2} - \sqrt{2\beta} = 0$, namely $\boxed{\beta = \frac{1}{8}}$.

2. (20 points) Consider the following scaler conservation law

$$u_t + (f(u))_x = 0, \quad f(u) = \begin{cases} (u+1)(8u+10) & u < -1 \\ 1 - u^2 & -1 \le u \le 1 \\ (u-1)(8u-10) & u > 1 \end{cases},$$

with initial condition

$$u(x, t=0) = g(x) = \begin{cases} \frac{5}{4} & x < 0, \\ -\frac{5}{4} & x > 0. \end{cases}$$

(a) Write down the weak formulation of the initial value problem. You can keep f and g in the expression without plugging in the values.

(b) Find the *explicit* entropy solution of the problem. *Note: the flux is not convex.*

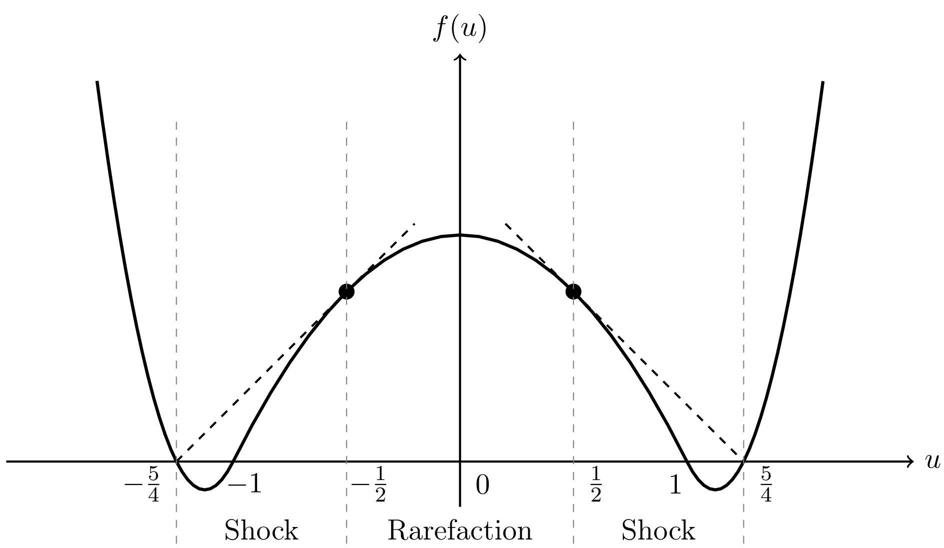
Solution: (a). The weak formulation of the equation states that, for any given test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$,

$$\boxed{\int_0^\infty \int_{\mathbb{R}} \big(u\phi_t + f(u)\phi_x\big)dxdt + \int_{\mathbb{R}} \phi(x,0)g(x)dx = 0.}$$

(b). We first compute the characteristic speed

$$f'(u) = \begin{cases} 16u + 18 & u < -1 \\ -2u & -1 \le u \le 1 \\ 16u - 18 & u > 1 \end{cases}, \text{ and initially } f'(g(x)) = \begin{cases} 2 & x < 0 \\ -2 & x > 0 \end{cases}.$$

As f is neither convex nor concave in $[-\frac{5}{4}, \frac{5}{4}]$, we need to use Oleynik condition to determine shocks.



As illustrated in the figure, the Oleynik condition is satisfied in $[-\frac{5}{4}, -\frac{1}{2}]$ and $[\frac{1}{2}, \frac{5}{4}]$. Therefore, there are two shocks with speed 1 and -1. In the range $[-\frac{1}{2}, \frac{1}{2}]$, since f is concave, there is a rarefaction wave. Inside the fan, $u(x,t) = (f')^{-1}(\frac{x}{t}) = -\frac{x}{2t}$. Note that $(f')^{-1}(y) = -\frac{y}{2}$.

To conclude, the entropy solution is

$$\boxed{u(x,t) = \begin{cases} \frac{5}{4} & x < -t \\ -\frac{x}{2t} & -t < x < t \\ -\frac{5}{4} & x > t \end{cases}.}$$

3. (20 points) Consider the initial value problem of the Klein-Gorden equation

$$\begin{cases} u_{tt} - \Delta u + m^2 u = 0, \quad x \in \mathbb{R}^n, \ t > 0 \\ u(x, t=0) = g(x), \quad u_t(x, t=0) = h(x), \end{cases}$$

where $m > 0$ is a constant.

(a) Find the wave speed $\left|\sigma/|y|\right|$ for any wave number y. Is the equation dispersive?

(b) Write down the definition of Fourier transform $\hat{u}(y,t)$, and solve $\hat{u}$.

(c) Show that there is at most one compactly supported classical solution of the problem. *Hint: use energy method.*

Solution: (a). Apply Ansatz $u(x,t) = e^{i(y\cdot x - \sigma t)}$ to the equation, we get

$$(-\sigma^2 + |y|^2 + m^2)u = 0,$$

which implies $|\sigma| = \sqrt{|y|^2 + m^2}$ and the wave speed for wave number y is $\boxed{\sqrt{1 + \frac{m^2}{|y|^2}}}$. Since the wave speed varies for different wave numbers, the equation is $\boxed{\text{dispersive}}$.

(b). The Fourier transform of u is defined as

$$\boxed{\hat{u}(y,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} e^{-ix\cdot y} u(y,t) dy.}$$

Under Fourier transform, the equation reads $\hat{u}_{tt} + (|y|^2 + m^2)\hat{u} = 0$, with initial condition $\hat{u}(y,0) = \hat{g}(y)$ and $\hat{u}_t(y,0) = \hat{h}(y)$. The solution is

$$\boxed{\hat{u}(y,t) = \hat{g}(y)\cos(\sqrt{|y|^2 + m^2}t) + \frac{\hat{h}(y)}{\sqrt{|y|^2 + m^2}} \sin(\sqrt{|y|^2 + m^2}t).}$$

(c). Suppose u_1 and u_2 are two classical solutions of the equation. Then, $w = u_1 - u_2$ also satisfies the equation with zero initial conditions. From (b), we know $\hat{w}(y,t) = 0$. Therefore, $w = 0$ and $u_1 = u_2$. This implies uniqueness.

Remark: one can also show that the energy $E(t) = \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2 + m^2 u^2) dx$ is conserved in time.

4. (15 points) Show that there is at most one classical solution to the initial-boundary value problem

$$\begin{cases} u_{tt} + cu_t - u_{xx} = f(x,t) & x \in (0,1), t \in (0,\infty), c \geq 0. \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & x \in (0,1),\ t = 0, \\ u(0,t) = u(1,t) = 0 & x = \{0,1\},\ t \in (0,\infty). \end{cases}$$

Solution: Let u_1, u_2 be two classical solutions of the initial-boundary value problem. Take $w = u_1 - u_2$. Then w satisfies

$$\begin{cases} w_{tt} + cw_t - w_{xx} = 0 & x \in (0,1), t \in (0,\infty), c \geq 0. \\ w(x,0) = 0, \quad w_t(x,0) = 0 & x \in (0,1),\ t = 0, \\ w(0,t) = w(1,t) = 0 & x = \{0,1\},\ t \in (0,\infty). \end{cases}$$

Multiply the equation by w_t and integrate in $\Omega[0,1]$, we obtain the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (w_t^2 + w_x^2)dx = -c\int_0^1 w_t^2 dx + (w_t w_x)|_0^1 \leq 0.$$

This implies the energy $E(t) = \int_0^1 (w_t^2 + w_x^2)dx$ is not increasing, i.e. $E(t) \leq E(0)$. From initial condition, we know $E(0) = 0$. Therefore, since $E(t)$ is non-negative, we conclude $E(t) = 0$, namely $w_t = w_x = 0$ almost everywhere. Since u, v are classical solutions, $w = u - v$ is smooth. So, w is a constant function. Since $w(x,0) = 0$, we get $w \equiv 0$ and $u \equiv v$. This implies uniqueness.

5. (20 points) Suppose a function G satisfies the following equation

$$-G'' + G = \delta(x), \quad -\infty < x < +\infty,$$
$$G(x), G'(x) \to 0 \text{ as } |x| \to \infty.,$$

where $\delta(x)$ is the Dirac delta distribution at $x = 0$.

(a) Write down the weak formulation of the equation.

(b) Prove that $G(x) = \frac{1}{2}e^{-|x|}$ is a weak solution of the equation.

(c) Write down a formula for the solution of

$$-u'' + u = f(x), \quad -\infty < x < +\infty.$$

Solution: (a). The weak formulation of the equation states that, for any given test function $\phi \in C_c^\infty(\mathbb{R})$,

$$\boxed{\int_{-\infty}^{\infty} \big(-\phi''(x)G(x) + \phi(x)G(x)\big)dx = \phi(0).}$$

(b). We verify that G satisfies the weak formulation.

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi''(x)G(x)dx &= \frac{1}{2}\int_{-\infty}^{0} \phi''(x)e^{x}dx + \frac{1}{2}\int_{0}^{\infty} \phi''(x)e^{-x}dx \\
&= \left[\frac{1}{2}\phi'(x)e^{x}\right]_{-\infty}^{0} - \frac{1}{2}\int_{-\infty}^{0} \phi'(x)e^{x}dx + \left[\frac{1}{2}\phi'(x)e^{-x}\right]_{0}^{\infty} + \frac{1}{2}\int_{0}^{\infty} \phi'(x)e^{-x}dx \\
&= \frac{1}{2}\phi'(0) - \frac{1}{2}\phi'(0) - \frac{1}{2}\int_{-\infty}^{0} \phi'(x)e^{x}dx + \frac{1}{2}\int_{0}^{\infty} \phi'(x)e^{-x}dx \\
&= -\left[\frac{1}{2}\phi(x)e^{x}\right]_{-\infty}^{0} + \frac{1}{2}\int_{-\infty}^{0} \phi(x)e^{x}dx + \left[\frac{1}{2}\phi(x)e^{-x}\right]_{0}^{\infty} + \int_{0}^{\infty} \frac{1}{2}\phi(x)e^{-x}dx \\
&= -\phi(0) + \int_{-\infty}^{\infty} \phi(x)G(x)dx.
\end{aligned}$$

So, G satisfies the weak formulation. Clearly, $G(x), G'(x) \to 0$ as $|x| \to \infty$.

(c). The solution is

$$\boxed{u(x) = G * f(x) = \int_{-\infty}^{\infty} G(x-y)f(y)dy.}$$

6. (10 points) Suppose u is smooth and solves the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.
   (a) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
   (b) Use (a) to show $v(x,t) := x \cdot \nabla u(x,t) + 2tu_t(x,t)$ solves the heat equation as well. *Hint: one can show that $\frac{d}{d\lambda}u_\lambda$ solves the heat equation.*

Solution: (a). Given any $\lambda \in \mathbb{R}$, $\partial_t u_\lambda(x,t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$, and $\Delta u_\lambda(x,t) = \lambda^2 \Delta u(\lambda x, \lambda^2 t)$. Therefore,

$$(u_t - \Delta u)(x,t) = \lambda^2((u_\lambda)_t - \Delta u_\lambda)(\lambda x, \lambda^2 t) = 0.$$

(b). Since u_λ solves the heat equation, $\frac{d}{d\lambda}u_\lambda$ also solves the heat equation, for any $\lambda \in \mathbb{R}$. Compute

$$\frac{d}{d\lambda}u_\lambda = x \cdot \nabla u(\lambda x, \lambda^2 t) + 2\lambda u_t(\lambda x, \lambda^2 t).$$

Clearly, $v = \frac{d}{dt}u_\lambda\big|_{\lambda=1}$. So, v also solves the heat equation.

7. (15 points) Let $B = \{x \in \mathbb{R}^3 : |x| < \pi\}$, and let u be smooth up to the boundary in B, $u = 0$ on the boundary of B. Let $\Delta u + u = f$. Prove that

$$\int_B \frac{\sin|x|}{|x|} f(x)dx = 0.$$

Hint: one can use spherical representation of Laplacian operator: in 3D, if u is radially symmetric, namely $u(x) = v(|x|) = v(r)$, then

$$\Delta u = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dv}{dr}\right).$$

Solution: Compute

$$\Delta \frac{\sin|x|}{|x|} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\frac{\sin r}{r}\right) = -\frac{\sin r}{r} = -\frac{\sin|x|}{|x|}.$$

Then, we use Gauss-Green formula and compute

$$\int_B \frac{\sin|x|}{|x|} f(x)dx = \int_B \frac{\sin|x|}{|x|}(\Delta u(x) + u(x))dx$$
$$= \int_{\partial B}\left[\frac{\sin|x|}{|x|}\frac{\partial u(x)}{\partial \mathbf{n}} - u(x)\frac{\partial}{\partial \mathbf{n}}\left(\frac{\sin|x|}{|x|}\right)\right] dS(x) + \int_B \Delta\frac{\sin|x|}{|x|}u(x)dx + \int_B \frac{\sin|x|}{|x|}u(x)dx$$

For the first term, $\frac{\sin|x|}{|x|} = \frac{\sin\pi}{\pi} = 0$ and $u(x) = 0$ when $x \in \partial B$. Therefore,

$$\int_B \frac{\sin|x|}{|x|} f(x)dx = 0 + \int_B \left(-\frac{\sin|x|}{|x|}\right) u(x)dx + \int_B \frac{\sin|x|}{|x|}u(x)dx = 0.$$