CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Sample Final Exam, Solution Name: ____________________________

1. (20 points) Find the explicit local solution for the following initial value problem

$$\begin{cases} u_t + \frac{(u_x)^2+x^2}{2} = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, t = 0) = \frac{x^2}{2}. \end{cases}$$

Solution: The equation has the form $F(x, t, z, p_1, p_2) = p_2 + (p_1^2 + x^2)/2 = 0$. The system of characteristic paths starting from $(\alpha, 0)$ reads

$$\begin{cases} \dot{x} = p_1 \\ \dot{t} = 1 \\ \dot{z} = p_1^2 + p_2 \\ \dot{p}_1 = -x \\ \dot{p}_2 = 0 \end{cases} \qquad \text{subject to initial condition} \qquad \begin{cases} x(0) = \alpha \\ t(0) = 0 \\ z(0) = \frac{\alpha^2}{2} \\ p_1(0) = \alpha \\ p_2(0) = -\alpha^2 \end{cases}$$

where $p_1(0)$ is computed from initial data, and $p_2(0)$ is computed from the equation.

To solve the ODE system, we first observe $t(s) = s$. Therefore, the parameter is simply t. Also $p_2(t) = -\alpha^2$. The coupled dynamics (x, p_1) yields $x(t) = \alpha(\cos t + \sin t)$ and $p_1(t) = \alpha(\cos t - \sin t)$. Plug in everything to the z equation and get $\dot{z} = -\alpha^2 \sin(2t)$, and therefore $z(t) = \frac{\alpha^2}{2}\cos(2t)$. The solution along the characteristic path starting from $(\alpha, 0)$ reads

$$u(x(t), t) = \frac{\alpha^2}{2}\cos(2t).$$

Now, we invert the map $(\alpha, t) \to (x, t)$, and get $\alpha = \frac{x}{\cos t + \sin t}$. Therefore, the solution of the initial value problem is

$$u(x, t) = \frac{x^2}{2(\cos t + \sin t)^2}\cos(2t) = \frac{x^2 \cos(2t)}{2(1 + \sin(2t))}.$$

Remark: From the expression, one can see that the solution exists for $t < 3\pi/4$.

2. (20 points) Consider the following equation

$$u_t + \left(\frac{u^2}{2} + u\right)_x = 0, \quad x \in \mathbb{R}, \ t \geq 0,$$

with initial condition

$$u(x, t = 0) = g(x) = \begin{cases} -2 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Write down the weak formulation of the initial value problem.

(b) Find the *explicit* entropy solution of the problem.

Solution: (a). The weak formulation of the equation states that, for any given test function $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$,

$$\int_0^\infty \int_{\mathbb{R}} \left[u\phi_t + \left(\frac{u^2}{2} + u\right)\phi_x \right] dxdt + \int_{\mathbb{R}} \phi(x,0)g(x)dx = 0.$$

Plug in the initial condition, we get

$$\int_0^\infty \int_{\mathbb{R}} \left[u\phi_t + \left(\frac{u^2}{2} + u\right)\phi_x \right] dxdt - 2\int_{-1}^0 \phi(x,0)dx = 0.$$

(b). The flux $F(u) = \frac{u^2}{2} + u$, and the wave speed $F'(u) = u+1$. Initially, we have

$$F'(u(x,0)) = \begin{cases} -1 & -1 < x < 0 \\ 1 & \text{otherwise} \end{cases}.$$

Since the flux is strictly convex ($F''(u) = 1 > 0$), using Lax condition, we know there is a shock wave at $x=-1$, and a rarefaction wave at $x=0$.

For the shock wave, the speed of the shock satisfies the Rankine-Hugoniot condition

$$\begin{cases} \dot\sigma = \frac{F(-2)-F(0)}{-2-0} = 0 \\ \sigma(0) = -1 \end{cases} \quad \Rightarrow \quad \sigma(t) = -1.$$

For the rarefaction wave, the fan is $\frac{x}{t} \in [-1,1]$. Inside the fan, $u(x,t) = (F')^{-1}(\frac{x}{t}) = \frac{x}{t} - 1$. Note that $(F')^{-1}(y) = y-1$.

The rarefaction fan touches the shock discontinuity at $t=1$. So before they meet, the entropy solution reads

$$u(x,t) = \begin{cases} 0 & x < -1 \\ -2 & -1 < x < -t \\ \frac{x}{t} - 1 & -t < x < t \\ 0 & x > t \end{cases}, \quad t \le 1.$$

After $t=1$, the shock interacts with the rarefaction fan. As $F'(u_-) = 1$ and $F'(u_+) \in (-1,1)$, the entropy solution admits a shock. The Rankine-Hugoniot condition says

$$\begin{cases} \dot\sigma = \frac{F(\frac{\sigma}{t}-1)-F(0)}{\frac{\sigma}{t}-1-0} = \frac{\sigma}{2t} + \frac{1}{2} \\ \sigma(1) = -1 \end{cases} \quad \Rightarrow \quad \sigma(t) = t - 2\sqrt{t}.$$

Therefore, the solution after time 1 reads

$$u(x,t) = \begin{cases} 0 & x < t - 2\sqrt{t} \\ \frac{x}{t} - 1 & t - 2\sqrt{t} < x < t \\ 0 & x > t \end{cases}, \quad t \ge 1.$$

Remark: drawing a picture is always helpful.

3. (10 points) Consider the following system of scaler conservation laws on (u, v)

$$\begin{cases} u_t - (q(v))_x = 0 \\ v_t - (p(u))_x = 0, \end{cases}$$

where p, q are C^1 functions with $p', q' > 0$. Show that the system is strictly hyperbolic.

Solution: We first write the system in vector form

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & q'(v) \\ p'(u) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0.$$

The eigenvalues of the matrix satisfies $\lambda^2 - p'(u)q'(v) = 0$. So, $\lambda = \pm\sqrt{p'(u)q'(v)}$. Since the eigenvalues are distinct real numbers, the system is strictly hyperbolic.

4. (15 points) Consider Airy's equation $u_t + u_{xxx} = 0$, with initial condition $u(x, 0) = g(x)$, for $x \in \mathbb{R}$.
   (a) Find the wave speed $\big|\sigma/|y|\big|$ for any wave number y. Is the equation dispersive?
   (b) Write down the definition of Fourier transform $\hat{u}(y, t)$, and solve $\hat{u}$.
   (c) Prove that the solution preserves L^2 norm in time: $\|u(\cdot, t)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$, for $t > 0$.

Solution: (a). Apply Ansatz $u(x, t) = e^{i(yx - \sigma t)}$ to the equation, we get

$$(i\sigma + (iy)^3)u = 0,$$

which implies $\sigma = -y^3$ and the wave speed for wave number y is $\boxed{y^2}$. Since the wave speed varies for different wave numbers, the equation is $\boxed{\text{dispersive}}$.

(b). The Fourier transform of u is defined as

$$\boxed{\hat{u}(y, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} u(y, t) dy.}$$

Under Fourier transform, the equation reads $\hat{u}_t + (iy)^3 u = 0$, with initial condition $\hat{u}(y, 0) = \hat{g}(y)$. The solution is $\boxed{\hat{u}(y, t) = e^{iy^3 t}\hat{g}(y)}$.

(c).

$$\|u(\cdot, t)\|_{L^2}^2 = \|\hat{u}(\cdot, t)\|_{L^2}^2 = \int_{\mathbb{R}} |e^{iy^3 t}\hat{g}(y)|^2 dy = \int_{\mathbb{R}} |\hat{g}(y)|^2 dy = \|\hat{g}\|_{L^2}^2 = \|g\|_{L^2}^2.$$

The first and last equality is due to Plancherel's theorem, and the middle equality is due to $|e^{iw}| = 1$ for all $w \in \mathbb{R}$.

5. (20 points) Find the *explicit* solution in the first quadrant $x > 0$ and $t > 0$ of the wave equation with initial-boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > 0, t > 0, \\ u(x,0) = g(x), & u_t(x,0) = h(x), \\ u_t(0,t) = a u_x(0,t), & a \neq -c, \end{cases}$$

where $g(x)$ and $h(x)$ are C^2 functions which vanish near $x = 0$. Show that no solution exists in general if $a = -c$.

Hint: The solution of 1D wave equation can be written as $u = F(x+ct) + G(x-ct)$. Determine $F(x)$ and $G(x)$ for $x \geq 0$ from the initial condition, and determine $G(x)$ for $x < 0$ from the boundary condition.

Solution: The general solution of the equation has the form $u = F(x + ct) + G(x - ct)$ where F, G are chosen to satisfy boundary conditions. Plug in the boundary conditions and get

$$\begin{cases} F(x) + G(x) = g(x) & x > 0 \\ cF'(x) - cG'(x) = h(x) & x > 0 \\ cF'(ct) - G'(-ct) = aF'(ct) + aG'(-ct) & t > 0. \end{cases}$$

From the first two equations, we solve for F and G for $x > 0$

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(s)ds, \quad G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(s)ds.$$

This yields the d'Alembert formula for $x - ct > 0$.

For $x - ct < 0$, we use the third equation and get

$$(a + c)G'(y) = (c - a)F'(-y), \quad y < 0.$$

If $a = -c$, there is no solution in general (unless $F' \equiv 0$). Otherwise, we get

$$G(y) = \frac{c-a}{a+c}F(-y) = \frac{c-a}{2(a+c)}\left[g(-y) + \frac{1}{c}\int_0^{-y} h(s)ds\right].$$

Put everything together, we get the final solution

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds & x \geq ct \\ \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(s)ds + \frac{c-a}{2(a+c)}\left[g(ct-x) + \frac{1}{c}\int_0^{ct-x} h(s)ds\right] & x < ct \end{cases}}$$

Note that for $x > ct$, the solution is given by d'Alembert formula, as the boundary condition is ineffective due to finite speed of propagation.

6. (15 points) Consider heat equation with a source term and a Dirichlet boundary condition

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0, +\infty) \\ u(x, 0) = g(x) & \text{for } x \in \Omega,\ t = 0 \\ u(x, t) = h(x, t) & \text{for } x \in \partial\Omega,\ t \in (0, \infty) \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Prove that there is at most one classical solution which solves the initial-boundary value problem.

Solution: Let u, v be two classical solutions of the initial-boundary value problem. Take $w = u - v$. Then w satisfies

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega \times (0, +\infty) \\ w(x, 0) = 0 & \text{for } x \in \Omega,\ t = 0 \\ w(x, t) = 0 & \text{for } x \in \partial\Omega,\ t \in (0, \infty) \end{cases}$$

Multiply the heat equation by w and integrate in Ω, we obtain the following energy estimate

$$E(t) := \frac{1}{2}\frac{d}{dt}\int_\Omega w(x,t)^2 dx = \int_\Omega w(x,t)\Delta w(x,t) dx = -\int_\Omega |Dw|^2 dx \leq 0,$$

where we use the zero Dirichlet boundary condition when performing integration by parts. This implies the energy is not increasing, i.e. $E(t) \leq E(0)$. From initial condition, we know $E(0) = 0$. Therefore, since $E(t)$ is non-negative, we conclude $E(t) = 0$, namely $w = 0$ almost everywhere. Since u, v are classical solutions, $w = u - v$ is continuous. Therefore, $w \equiv 0$ and $u \equiv v$. This implies uniqueness.

Remark: one can also use maximum principle to prove uniqueness.

7. (20 points) Let u be a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B_1(0) \\ u = g & \text{on } \partial B_1(0). \end{cases},$$

where f and g are bounded functions in $B_1(0)$ and $\partial B_1(0)$, respectively.

(a) We say $v \in C^2(\bar{\Omega})$ is *subharmonic* if $-\Delta v \leq 0$. Prove the mean value theorem for subharmonic v that

$$v(x) \leq \fint_{B_r(x)} v(y) dy \quad \text{for all } B_r(x) \in \Omega.$$

(b) Prove that if v is subharmonic, then its satisfies the maximum principle

$$\max_{\bar{\Omega}} v = \max_{\partial\Omega} v.$$

(c) Let $v(x) = u(x) + \frac{M}{2n}|x|^2$, where $M = \max_{|x|\leq 1} |f(x)|$. Prove that v is subharmonic in $B_1(0)$.

(d) Prove that there exists a constant C, depending only on dimension n, such that

$$\max_{B_1(0)} |u| \leq C \left(\max_{\partial B_1(0)} |g| + \max_{B_1(0)} |f| \right).$$

Hint: apply maximum principle on v in (c).

Solution: (a). The proof is similar to the mean value theorem for harmonic functions.

First, let $\phi(r) = ⨍_{\partial B_r(x)} v(y) dS(y)$. Then, clearly $\lim_{r \to 0+} \phi(r) = v(x)$. Compute

$$\begin{aligned} \phi'(r) =& \frac{d}{dr} ⨍_{\partial B_1(0)} v(x + rz) dS(z) = ⨍_{\partial B_1(0)} z \cdot \nabla v(x + rz) dS(z) \\ =& ⨍_{\partial B_r(x)} \frac{\partial}{\partial \mathbf{n}} v(y) dS(y) = \frac{1}{|\partial B_r(x)|} \int_{B_r(x)} \Delta v(y) dy \geq 0. \end{aligned}$$

Therefore, for all $r > 0$, $\phi(r) \geq \phi(0) = v(x)$.

Next, we write

$$\begin{aligned} \int_{B_r(x)} v(y) dy = \int_0^r \int_{\partial B_\rho(x)} v(y) dS(y) d\rho = \int_0^r |\partial B_\rho(x)| \phi'(\rho) d\rho \\ \geq \phi(0) \int_0^r |\partial B_\rho(x)| d\rho = v(x) |B_r(x)|. \end{aligned}$$

This directly implies the inequality.

(b). The proof is similar to the maximum principle for harmonic functions.

Let A be a subset of Ω defined as

$$A = \{x \in \Omega \mid v(x) = L\}, \quad \text{where} \quad L = \max_{y \in \bar{\Omega}} v(y).$$

As v is continuous, clearly $A = v^{-1}(\{L\})$ is a close set. To prove A is open, take any $x \in \Omega$. Pick $r = \frac{1}{2} \text{dist}(x, \partial\Omega)$, then $B_r(x) \subset \Omega$. Applying (a), we get

$$L = v(x) \leq ⨍_{B_r(x)} v(y) dy \leq ⨍_{B_r(x)} L dy = L.$$

For the second inequality, the equality is attained only when $v(y) = L$ for all $y \in B_r(x)$. Therefore, $B_r(x) \in A$. A is open.

Since A is both open and close, then $A = \emptyset$ or $A = \Omega$. In both cases, the maximum is attained at the boundary.

(c). Let $v(x) = u(x) + \frac{M}{2n} |x|^2$. Compute $-\Delta v = -f - M \leq 0$. So v is subharmonic.

(d). By strong maximum principle, we get

$$\max_{x \in B_1(0)} u(x) \leq \max_{x \in B_1(0)} v(x) = \max_{x \in \partial B_1(0)} v(x) = \max_{\partial B_1(0)} g(x) + \frac{M}{2n}.$$

On the other hand, define $w(x) = -u(x) + \frac{M}{2n}|x|^2$. Compute $-\Delta w = f - M \le 0$. So w is also subharmonic. By maximum principle (b), we get

$$\max_{x\in B_1(0)} -u(x) \le \max_{x\in B_1(0)} w(x) = \max_{x\in\partial B_1(0)} w(x) = \max_{\partial B_1(0)} (-g(x)) + \frac{M}{2n}.$$

Put the two estimates together, we conclude

$$\max_{x\in B_1(0)} |u(x)| \le \max_{\partial B_1(0)} |g(x)| + \frac{1}{2n} \max_{B_1(0)} |f|.$$

The inequality is proved with $C = 1$.