CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Final Exam Name: ______________________

Instruction: (READ CAREFULLY!!)
Show all work clearly and in order. Textbook (Evans), your own notes and your own homework assignments are allowed during the exam. Use **no** other books, calculators, cellphones, communication with others, computers, etc. You are required to sign the *honor pledge*.

For CAAM423/MATH423 students: The exam has a total of 105 points. You are required to finish problems 1-6. Use **two hundred** consecutive minutes of your choice to finish the exam. If you get more than 100 points, your final score will be 100.

For CAAM523/MATH513 students: The exam has a total of 120 points. You are required to finish problems 1-7. Use **four** consecutive hours of your choice to finish the exam.

Please **print and fill** this page, and attach it with your submission. Turn in your exam on Wednesday December 13 at 10am-12pm or 2pm-4pm to my office HBH 426.

Starting time: ______________________ Ending time: ______________________

Honor Pledge: The Rice University Honor Pledge reads:

"On my honor, I have neither given nor received any unauthorized aid on this exam."

Please write the exact wording of the Pledge, following by your signature, in the space below:

Pledge: ____

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Your Signature: ____

Question:	1	2	3	4	5	6	7	Total
Points:	20	20	20	15	20	10	15	120
Score:								

- Good Luck -

1. (20 points) Consider the following initial value problem

$$\begin{cases} (u_{x_1})^2 + \frac{1}{2}(u_{x_2})^2 = \beta + x_1^2, & \text{in } \mathbb{R}^2, \\ u = \frac{x_1^2}{2}. & \text{on } \mathbb{R} \times \{x_2 = 0\}, \end{cases}$$

   (a) Find the *explicit* solution $u(x_1, x_2)$ of this problem for any given $\beta > 0$.

   (b) For what value(s) of β is $u \equiv 0$ on the parabola $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1^2\}$?

2. (20 points) Consider the following scaler conservation law

$$u_t + (f(u))_x = 0, \quad f(u) = \begin{cases} (u+1)(8u+10) & u < -1 \\ 1 - u^2 & -1 \le u \le 1 \\ (u-1)(8u-10) & u > 1 \end{cases},$$

   with initial condition

$$u(x, t=0) = g(x) = \begin{cases} \frac{5}{4} & x < 0, \\ -\frac{5}{4} & x > 0. \end{cases}$$

   (a) Write down the weak formulation of the initial value problem. You can keep f and g in the expression without plugging in the values.

   (b) Find the *explicit* entropy solution of the problem. *Note: the flux is not convex.*

3. (20 points) Consider the initial value problem of the Klein-Gorden equation

$$\begin{cases} u_{tt} - \Delta u + m^2 u = 0, & x \in \mathbb{R}^n, \ t > 0 \\ u(x, t=0) = g(x), & u_t(x, t=0) = h(x), \end{cases}$$

   where $m > 0$ is a constant.

   (a) Find the wave speed $\big|\sigma/|y|\big|$ for any wave number y. Is the equation dispersive?

   (b) Write down the definition of Fourier transform $\hat{u}(y, t)$, and solve $\hat{u}$.

   (c) Show that there is at most one compactly supported classical solution of the problem. *Hint: use energy method.*

4. (15 points) Show that there is at most one classical solution to the initial-boundary value problem

$$\begin{cases} u_{tt} + cu_t - u_{xx} = f(x, t) & x \in (0,1), t \in (0, \infty), c \ge 0. \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & x \in (0,1), \ t = 0, \\ u(0, t) = u(1, t) = 0 & x = \{0, 1\}, \ t \in (0, \infty). \end{cases}$$

5. (20 points) Suppose a function G satisfies the following equation

$$-G'' + G = \delta(x), \quad -\infty < x < +\infty,$$
$$G(x), G'(x) \to 0 \text{ as } |x| \to \infty.,$$

   where $\delta(x)$ is the Dirac delta distribution at $x = 0$.

   (a) Write down the weak formulation of the equation.

(b) Prove that $G(x) = \frac{1}{2}e^{-|x|}$ is a weak solution of the equation.

(c) Write down a formula for the solution of

$$-u'' + u = f(x), \quad -\infty < x < +\infty.$$

6. (10 points) Suppose u is smooth and solves the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

   (a) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

   (b) Use (a) to show $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well. *Hint: one can show that $\frac{d}{d\lambda}u_\lambda$ solves the heat equation.*

7. (15 points) Let $B = \{x \in \mathbb{R}^3 : |x| < \pi\}$, and let u be smooth up to the boundary in B, $u = 0$ on the boundary of B. Let $\Delta u + u = f$. Prove that

$$\int_B \frac{\sin|x|}{|x|} f(x) dx = 0.$$

Hint: one can use spherical representation of Laplacian operator: in 3D, if u is radially symmetric, namely $u(x) = v(|x|) = v(r)$, then

$$\Delta u = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dv}{dr}\right).$$