CAAM/MATH423 Partial Differential Equations I — Fall 2015

Midterm Exam — **Solution**

1. (25 points) Consider the equation

$$-(u_{x_1})^2 + (u_{x_2})^2 + x_2^2 = 0, \quad x_1 \in \mathbb{R},\ x_2 > 0$$

with initial condition $u(x_1, 0) = g(x_1)$, where $g(x_1) \in C^1$ is strictly increasing. Also assume $u_{x_2}(x_1, 0) \geq 0$.

(a) Find explicitly the characteristics $x_1(\alpha, s)$ and $x_2(\alpha, s)$ starting at the point $(\alpha, 0)$.

Solution: The equation has the form $F(x_1, x_2, z, p_1, p_2) = -p_1^2 + p_2^2 + x_2^2 = 0$. The system of characteristic paths starting from $(\alpha, 0)$ reads

$$\begin{cases} \dot{x}_1 = -2p_1 \\ \dot{x}_2 = 2p_2 \\ \dot{z} = -2p_1^2 + 2p_2^2 \\ \dot{p}_1 = 0 \\ \dot{p}_2 = -2x_2 \end{cases} \qquad \text{subject to initial condition} \qquad \begin{cases} x_1(0) = \alpha \\ x_2(0) = 0 \\ z(0) = g(\alpha) \\ p_1(0) = g'(\alpha) \\ p_2(0) = g'(\alpha) \end{cases}$$

where $p_1(0)$ is computed from initial data, and $p_2(0)$ is computed from the equation. To solve the ODE system, we first solve $p_1(s) = g'(\alpha)$, and consequently,

$$\boxed{x_1(s) = \alpha - 2g'(\alpha)s}.$$

(x_2, p_2) forms a coupled system. Equivalently, we can write it as a second order ODE for x_2 as follows

$$\ddot{x}_2 = -4x_2, \quad \text{subject to} \quad x_2(0) = 0, \quad \dot{x}_2(0) = 2g'(\alpha).$$

The solution is given by $\boxed{x_2(s) = g'(\alpha)\sin(2s)}$ and therefore $p_2(s) = g'(\alpha)\cos(2s)$.

(b) In the case when $g(x_1) = x_1$, sketch the characteristics and write down an explicit solution. *Hint: To save you some time, use directly* $\int_0^t \sin^2(2s)ds = \frac{t}{2} - \frac{1}{8}\sin(4t)$. *Note that you only have to run* $s \in [0, \pi/4]$ *to construct a solution.*

Solution: In the case $g(x_1) = x_1$, we get $x_1(s) = \alpha - 2s$, $x_2(s) = \sin(2s)$, $p_1(s) = 1$ and $p_2(s) = \cos(2s)$. The dynamics of z then reads

$$\dot{z} = -2 + 2\cos^2(2s) = -2\sin^2(2s), \quad \text{subject to} \quad z(0) = \alpha.$$

Therefore, $z(s) = \alpha - 2\int_0^s \sin^2(2\tau)d\tau = \alpha - s + \frac{1}{4}\sin(4s)$. Now, we invert the map $(\alpha, s) \to (x_1, x_2)$ and get $\alpha = x_1 + \arcsin(x_2)$, $s = \frac{1}{2}\arcsin(x_2)$. Finally, we reach a solution

$$\boxed{u(x_1, x_2) = z(\alpha, s) = x_1 + \frac{1}{2}\arcsin(x_2) + \frac{1}{2}x_2\sqrt{1 - x_2^2}.}$$

2. (20 points) Consider Airy's equation $u_t + u_{xxx} = 0$, with initial condition $u(x,0) = g(x)$, for $x \in \mathbb{R}$.

   (a) Find the wave speed $|\sigma/|y||$ for any wave number y. Is the equation dispersive?

   Solution: Apply Ansatz $u(x,t) = e^{i(yx-\sigma t)}$ to the equation, we get

   $$(i\sigma + (iy)^3)u = 0,$$

   which implies $\sigma = -y^3$ and the wave speed for wave number y is $\boxed{y^2}$. Since the wave speed varies for different wave numbers, the equation is $\boxed{\text{dispersive}}$.

   (b) Write down the definition of Fourier transform $\hat{u}(y,t)$, and solve $\hat{u}$.

   Solution: The Fourier transform of u is defined as

   $$\boxed{\hat{u}(y,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} u(y,t)dy.}$$

   Under Fourier transform, the equation reads $\hat{u}_t + (iy)^3 u = 0$, with initial condition $\hat{u}(y,0) = \hat{g}(y)$. The solution is $\boxed{\hat{u}(y,t) = e^{iy^3 t}\hat{g}(y)}$.

   (c) Prove that the solution preserves L^2 norm in time: $\|u(\cdot,t)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$, for $t > 0$.

   Solution:

   $$\|u(\cdot,t)\|_{L^2}^2 = \|\hat{u}(\cdot,t)\|_{L^2}^2 = \int_{\mathbb{R}} |e^{iy^3 t}\hat{g}(y)|^2 dy = \int_{\mathbb{R}} |\hat{g}(y)|^2 dy = \|\hat{g}\|_{L^2}^2 = \|g\|_{L^2}^2.$$

   The first and last equality is due to Plancherel's theorem, and the middle equality is due to $|e^{iw}| = 1$ for all $w \in \mathbb{R}$.

3. (25 points) Find the *explicit* solution in the first quadrant $x > 0$ and $t > 0$ of the wave equation with initial-boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad x > 0, t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \\ u_t(0,t) = a u_x(0,t), \quad a \neq -c, \end{cases}$$

where $g(x)$ and $h(x)$ are C^2 functions which vanish near $x = 0$. Show that no solution exists in general if $a = -c$.

Hint: The solution of 1D wave equation can be written as $u = F(x+ct) + G(x-ct)$. Determine $F(x)$ and $G(x)$ for $x \geq 0$ from the initial condition, and determine $G(x)$ for $x < 0$ from the boundary condition.

Solution: The general solution of the equation has the form $u = F(x+ct) + G(x-ct)$ where F, G are chosen to satisfy boundary conditions. Plug in the boundary conditions and get

$$\begin{cases} F(x) + G(x) = g(x) & x > 0 \\ cF'(x) - cG'(x) = h(x) & x > 0 \\ cF'(ct) - G'(-ct) = aF'(ct) + aG'(-ct) & t > 0. \end{cases}$$

From the first two equations, we solve for F and G for $x > 0$

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(s)ds, \quad G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(s)ds.$$

This yields the d'Alembert formula for $x - ct > 0$.

For $x - ct < 0$, we use the third equation and get

$$(a+c)G'(y) = (c-a)F'(-y), \quad y < 0.$$

If $a = -c$, there is no solution in general (unless $F' \equiv 0$). Otherwise, we get

$$G(y) = \frac{c-a}{a+c}F(-y) = \frac{c-a}{2(a+c)}\left[g(-y) + \frac{1}{c}\int_0^{-y} h(s)ds\right].$$

Put everything together, we get the final solution

$$u(x,t) = \begin{cases} \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds & x \geq ct \\ \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(s)ds + \frac{c-a}{2(a+c)}\left[g(ct-x) + \frac{1}{c}\int_0^{ct-x} h(s)ds\right] & x < ct \end{cases}$$

Note that for $x > ct$, the solution is given by d'Alembert formula, as the boundary condition is ineffective due to finite speed of propagation.

4. (15 points) Consider Poisson equation with Robin boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}}(x) + \alpha(x) u(x) = g(x) & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth, bounded domain. α is a continuous positive functions on $\partial\Omega$. Prove that there is at most one classical solution which solves the boundary value problem.

Solution: Suppose u and v are two classical solutions of the boundary value problem, then $w := u - v$ solves the following boundary value problem

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \mathbf{n}}(x) + \alpha(x) w(x) = 0 & \text{on } \partial\Omega \end{cases}$$

Take energy $E = \int_\Omega |Dw(x)|^2 dx$ and apply Gauss-Green formula

$$\begin{aligned} E &= \int_\Omega Dw \cdot Dw dx = -\int_\Omega w \Delta w dx + \int_{\partial\Omega} w \frac{\partial w}{\partial \mathbf{n}} dS(x) \\ &= 0 + \int_{\partial\Omega} w(-\alpha w) dS(x) = -\int_{\partial\Omega} \alpha w^2 dS(x) \leq 0. \end{aligned}$$

Therefore, $E = 0$, which implies that w is a constant in $\bar{\Omega}$. Moreover, take $x \in \partial\Omega$, the boundary condition tells $0 + \alpha(x) w(x) = 0$. Therefore, $w \equiv 0$, i.e $u \equiv v$. So there is at most one classical solution.

5. (25 points) Consider the following 1D equation

$$-u'' + u = f(x), \quad -\infty < x < \infty,$$

where f is a bounded smooth function in $\mathbb{R}$.

(a) Define $\Phi(x) = \frac{1}{2}e^{-|x|}$. Verify that $-\Phi''(x) + \Phi(x) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$. (Note that at $x = 0$, Φ'' is not well-defined. Φ is called the fundamental solution for $-u'' + u = 0$.)

Solution: For $x > 0$, $\Phi(x) = \frac{1}{2}e^{-x}$. Compute

$$-\Phi''(x) + \Phi(x) = -\frac{1}{2}(-1)^2 e^{-x} + \frac{1}{2}e^{-x} = 0.$$

Similarly, we can check for the case $x < 0$.

(b) Define $u = \Phi * f$. Prove that u is a smooth solution of the equation.

Hint: To verify u satisfies the equation, one can write

$$u''(x) = \int_{-\infty}^{\infty} \Phi(y) f''(x-y)dy = \int_{-\infty}^{0} \frac{1}{2}e^{y} f''(x-y)dy + \int_{0}^{\infty} \frac{1}{2}e^{-y} f''(x-y)dy.$$

Perform integration by parts twice for each terms. The procedure is similar to but much easier than Poisson equation, not only because the equation is in 1D, but also because the fundamental solution is not singular (only not differentiable at origin).

Solution: We verify that u satisfies the equation

$$\begin{aligned}
u''(x) &= \int_{-\infty}^{\infty} \Phi(y) f''(x-y)dy = \int_{-\infty}^{0} \frac{1}{2}e^{y} f''(x-y)dy + \int_{0}^{\infty} \frac{1}{2}e^{-y} f''(x-y)dy \\
&= -\left[\frac{1}{2}e^{y} f'(x-y)\right]_{y=-\infty}^{0} + \int_{-\infty}^{0} \frac{1}{2}e^{y} f'(x-y)dy \\
&\quad - \left[\frac{1}{2}e^{-y} f'(x-y)\right]_{y=0}^{\infty} - \int_{0}^{\infty} \frac{1}{2}e^{-y} f'(x-y)dy \\
&= -\frac{1}{2}f'(x) - \left[\frac{1}{2}e^{y} f(x-y)\right]_{y=-\infty}^{0} + \int_{-\infty}^{0} \frac{1}{2}e^{y} f(x-y)dy \\
&\quad + \frac{1}{2}f'(x) + \left[\frac{1}{2}e^{-y} f(x-y)\right]_{y=0}^{\infty} + \int_{0}^{\infty} \frac{1}{2}e^{-y} f(x-y)dy \\
&= -\frac{1}{2}f(x) + \int_{-\infty}^{0} \Phi(y) f(x-y)dy - \frac{1}{2}f(x) + \int_{0}^{\infty} \Phi(y) f(x-y)dy = -f(x) + u(x).
\end{aligned}$$

Also, u is as smooth as f, namely

$$|D^\alpha u(x)| = |\Phi * (D^\alpha f)| \le \int_{\mathbb{R}} \Phi(y)|D^\alpha f(x-y)|dy \le \|D^\alpha f\|_{L^\infty} \int_{\mathbb{R}} \Phi(y)dy = \|D^\alpha f\|_{L^\infty} < \infty.$$