CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 9, Due on Tuesday, November 3, 2015

1. *(Viscous Burgers equation)* Consider the initial value problem of viscous Burgers equation

$$\begin{cases} \theta_t + \theta\theta_x = \nu\theta_{xx} & x \in \mathbb{R}, \quad t > 0 \\ \theta(x,0) = g(x) & x \in \mathbb{R}, \quad t = 0 \end{cases}$$

where $\nu > 0$ is the viscosity coefficient, and $g(x) \geq 0$.

a). Let $\Theta(x,t) = \int_{-\infty}^{x} \theta(y,t)dy$. Prove that Θ satisfies the following equation:

$$\begin{cases} \Theta_t + \frac{1}{2}(\Theta_x)^2 = \nu\Theta_{xx} & x \in \mathbb{R}, \quad t > 0 \\ \Theta(x,0) = G(x) := \int_{-\infty}^{x} g(y)dy & x \in \mathbb{R}, \quad t = 0 \end{cases}$$

b). Let $u(x,t) = \exp[-\Theta(x,t)/2\nu]$, prove that u satisfies the heat equation $u_t - \nu u_{xx} = 0$. What is the corresponding initial condition for u?

c). Solve u and use the relation to get an explicit solution θ. It should have the following expression:

$$\theta(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} g(y) \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}G(y)\right] dy}{\displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}G(y)\right] dy}.$$

Note: Unlike the inviscid Burgers equation, which has shock formation in finite time, the solution for the viscous equation is smooth. It indicates the regularizing effect of the diffusion term.

2. *(Non-existence of weak derivative)* Let $u(x) = \begin{cases} 0 & x \in [-1,0) \\ 1 & x \in [0,1] \end{cases}$. Prove that there exists no function $v \in L^\infty$ such that $u_x = v$ in the weak sense.
Hint: We prove the statement by contradiction. Suppose $\exists v \in L^\infty$ such that for all test functions $\phi \in C_0^\infty([-1,1])$. Write down the definition of weak derivatives. Pick a sequence of smooth function $\{\phi_m\}_{m=1}^\infty$, such that $\phi_m(0) = 1$, and supp$(\phi_m) = (-1/m, 1/m)$. Prove a contradiction when m is big enough.

3. *(Sobolev space)* Let $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$, where $x \in B(0,1) \subset \mathbb{R}^n$.

a). Explain that $u(x) \notin L^\infty(B(0,1))$.

b). Verify that $u(x) \in W^{1,n}(B(0,1))$.

Note: This equation is a counter example showing that the break down of Sobolev embedding in the critical case.