CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 7, Due on Thursday, October 15, 2015

1. *(Poisson's formula on a half ball)*

a). Read the construction and proof of on Poisson's formula for ball (page 39-41 in Evans book).

b). Consider half ball $U^+ := \{x \in \mathbb{R}^n \ : \ |x| < 1, \ x_n > 0\}$. The boundary value problem states

$$\begin{cases} \Delta u = 0 & \text{in } U^+ \\ u = g_1 & \text{on } \partial B(0,1) \cap \{x_n > 0\} \\ u = g_2 & \text{on } B(0,1) \cap \{x_n = 0\} \end{cases}$$

where the two boundary values g_1 and g_2 are set to be compatible when they intersect.

Verify that a Green's function can be expressed as

$$G(x,y) := \Phi(y-x) - \Phi(y-\hat{x}) - \Phi(|x|(y-\tilde{x})) + \Phi(|x|(y-\tilde{\hat{x}})),$$

where $\hat{x} = (x_1, \cdot, x_{n-1}, -x_n)$, and $\tilde{x} = x/|x|^2$.

Note: one has to check the corresponding $\phi^x(y)$ satisfies the boundary value problem. Use the result for half-space and ball directly.

c). Write down the Poisson's formula for Laplace equation on U^+. (You do not need to verify that the solution indeed solves the boudary value problem. But keep in mind that the verification is necessary, and the procedure is similar to the half-space and ball cases.)

2. *(Harnack's inequality in a ball)* Finish excercise 7 in section 2.5 of Evans book. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B(0,r)$.

3. *(Subharmonic function)* Finish excercise 5 in section 2.5 of Evans book. We say $v \in C^2(\bar{\Omega})$ is *subharmonic* if

$$-\Delta v \le 0.$$

a). Prove the mean value theorem for subharmonic v that

$$v(x) \le ⨍_{B(x,r)} v(y)dy \quad \text{for all } B(x,r) \in \Omega.$$

b). Prove the maximum principle $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$.

c). Let $\phi : \ \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

d). Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.