CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 4, Due on Tuesday, September 22, 2015

1. *(Parallelogram identity)* Suppose u is a classical solution of $u_{tt} - c^2 u_{xx} = 0$. Then, for any four points that have the forms $A(x,t), B(x+ac, t-a), C(x+(a-b)c, t-a-b), D(x-bc, t-b)$, such that the parallelogram ABCD lies inside the domain, we have the following identity

$$u(A) + u(C) = u(B) + u(D).$$

Hint: Take $\xi = x + ct, \eta = x - ct$. Change the coordinates from (x,t) to (ξ, η). The equation becomes $u_{\xi\eta} = 0$ (consult last homework), and the parallelogram becomes a rectangle under the new coordinates. It would be easy to check the identity then.

2. *(Yet another way to get d'Alembert formula)* We have derived the formula which solves 1D nonhomogenuous wave equation $u_{tt} - c^2 u_{xx} = f$, with zero initial data $u(x,0) = u_t(x,0) \equiv 0$. It reads

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau) dy d\tau.$$

Use this formula to derive d'Alembert formula.
Hint: Let u be the solution of 1D wave equation $u_{tt} - c^2 u_{xx} = 0$, subject to initial condition $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$. Take $v(x,t) = u(x,t) - g(x) - th(x)$ and apply the formula above.

3. *(Stokes' rule)* Finish Problem 18 in section 2.5 of Evans book. Assume u solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0, \; u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Show that $v := u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = h, \; v_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$