CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 13, Due on Thursday, December 3, 2015

1. *(Non-uniqueness of weak solution)* Consider the following Riemann problem for Burgers equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}, \quad \text{where} \quad g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

a). Write down the definition of the weak solution.

b). Find the shock solution which serves as a weak solution of the problem.

c). Prove that the following solution is also a weak solution:

$$u(x) = \begin{cases} 0 & \text{if } x < \frac{t}{4} \\ \frac{1}{2} & \text{if } \frac{t}{4} < x < \frac{3t}{4} \\ 1 & \text{if } x > \frac{3t}{4} \end{cases}$$

Hint: the solution contains two shocks. One should check that Rankine-Hugoniot condition is satisfied for both interfaces.

d). *(Optinal)* Construct another weak solution which contains two shocks.

e). Use the definition to prove that the following rarefaction solution is a weak solution of the problem:

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1 & \text{if } x > t \end{cases}.$$

Remark: Since the rarefaction solution is continuous, it satisfies Lax entropy condition. Hence, it is the unique entropy solution of the Riemann problem.

2. *(Entropy solution for general initial data)* Finish Exercise 20 in section 3.6 of Evans book. Compute explicitely the unique entropy solution of

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

Draw a picture documenting your answer, being sure to illustrate what happens for all time $t > 0$.