CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 11, Due on Tuesday, November 17, 2015

1. *(Divergence versus non-divergence forms)* Consider elliptic operator

$$\mathcal{L}u = -\sum_{i,j} a^{ij}(x)u_{x_i x_j}(x) + \sum_i b^i(x)u_{x_i}(x) + c(x)u(x),$$

with $a^{ij} \in C^1$ and $b^i, c \in L^\infty$. Rewrite $\mathcal{L}$ in divergence form.
Remark: The problem shows non-divergence form can be converted into divergence form, which one can then apply the theory on weak solutions. However, the equation in non-divergence form are in general more difficult to deal with, when a^{ij} is only assumed to be L^∞.

2. *(Mixed boundary value problem)* Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where Γ_0 and Γ_1 each have positive surface area and a smooth boundary. Consider the boundary value problem

$$\begin{cases} -\mathrm{div}(A(x)Du) + c(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \mathbf{n} \cdot (A(x)Du) = 0 & \text{on } \Gamma_1, \end{cases}$$

where A, c, f are smooth over $\bar{\Omega}$, $\mathbf{n}$ is the outward unit normal on $\partial\Omega$, c is positive, and the $n \times n$ matrix-valued function A is symmetric and satisfies the uniformly ellipticity condition: $\xi^T A(x)\xi \geq \theta|\xi|^2$, for all $\xi \in \mathbb{R}^n$ and $x \in \bar{\Omega}$. Give a weak formulation of this problem, determine the appropriate space H to work with, and use the Lax-Milgram theorem to show the existence of a weak solution in H.

3. *(Biharmonic equation)* Finish exercise 3 in section 6.6 of Evans book. A function $u \in H_0^2(\Omega)$ is a weak solution of this boundary-value problem for the *biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

provided

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx$$

for all test functions $v \in H_0^2(\Omega)$.

a). Prove that if $u \in C^4(\bar{\Omega})$ is a classical solution of the equation, u is also a weak solution.

b). Given $f \in H^{-2}(\Omega)$, prove that there exists a unique weak solution of the equation.