CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Final Exam Name: ____________________

Instruction: (READ CAREFULLY!!) Show all work clearly and in order. The exam has a total of 110 points. At most 100 points will be recorded. Use **three** consecutive hours of your choice to finish the exam. You may choose to extend for one hour (so a total of *four* consecutive hours) to finish. If so, 10 points will be deducted from your total grades. **State** your starting and ending time below. Textbook (Evans), your own notes and your own homework assignments are allowed during the exam. Use **no** other books, calculators, cellphones, communication with others, computers, etc. You are required to sign the *honor pledge*. Please hand in the exam on December 7 between 10am-12pm, or 2pm-4pm, to my office in HBH 328. Please attach this page with your name and signature on submission.

Starting time: ____________________ Ending time: ____________________

Honor Pledge: The Rice University Honor Pledge reads:

> "On my honor, I have neither given nor received any unauthorized aid on this exam."

Please write the exact wording of the Pledge, following by your signature, in the space below:

Pledge: ____

____

Your Signature: ____

Question:	1	2	3	4	5	6	7	Total
Points:	20	15	15	10	10	20	20	110
Score:								

1. (20 points) Find the explicit local solution for the following initial value problem

$$\begin{cases} u_t + \frac{(u_x)^2 + x^2}{2} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, t = 0) = g(x). \end{cases}$$

2. (15 points) Let $h \in C_0^\infty(\mathbb{R}^n)$, and let u be the solution to

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x, t = 0) = 0, & u_t(x, t = 0) = h(x). \end{cases}$$

Let $v(x,t) = \int_0^t u(x,s)ds$. Prove

$$\int_{\mathbb{R}^n} (\Delta v(x,t))^2 dx \le 4\|h\|_{L^2}^2.$$

Hint: Use Fourier transform in x. First solve $\hat{u}(y,t)$ and $\hat{v}(y,t)$. Then use Plancherel's theorem.

3. (15 points) Prove that there exists a constant C, depending only on dimension n, such that

$$\max_{B(0,1)} |u| \le C \left(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right)$$

whenever u is a smooth solution of

$$\begin{cases} \Delta u = f & \text{in } B(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Hint: Let $M = \max_{B(0,1)} |f|$. Define $v(x) = u(x) + \frac{M}{n}|x|^2$. Prove that v is subharmonic. Then apply maximum principle to v.

4. (10 points) Consider heat equation with a source term and a Dirichlet boundary condition

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0, +\infty) \\ u(x, 0) = g(x) & \text{for } x \in \Omega,\ t = 0 \\ u(x,t) = h(x,t) & \text{for } x \in \partial\Omega,\ t \in (0, \infty) \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Prove that there is at most one classical solution which solves the initial-boundary value problem.

5. (10 points) Let Ω be a bounded open set in $\mathbb{R}^3$. Suppose $u \in H^1(\Omega)$. Prove that $u^3 \in L^2(\Omega)$.

6. (20 points) Let Ω be a smooth bounded open domain in $\mathbb{R}^2$, and f is a continuous function in $\bar{\Omega}$. Find the weak formulation of the equation

$$-\Delta u + \frac{u}{3} + u_x = f$$

where $(x, y) \in \Omega$ with zero Dirichlet boundary condition. Show that the weak problem has a unique solution.

7. (20 points) Find the explicit entropy solution of the following equation

$$u_t + \left(\frac{u^2}{2} + u \right)_x = 0, \quad x \in \mathbb{R},\ t \ge 0,$$

with initial condition

$$u(x, t = 0) = g(x) = \begin{cases} -2 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$