MATH322 Introduction to Mathematical Analysis II — Spring 2016

Midterm Exam — **Solution**

1. (20 points) Let $f \in \mathcal{C}^\infty([a,b])$. Fix an $\epsilon > 0$, and set $\Gamma(f,\epsilon) = \{x \in [a,b] : |f(x)| < \epsilon\}$. Assume $\Gamma(f,\epsilon)$ has the form

$$\Gamma(f,\epsilon) = \bigcup_{i=1}^{n} (a_i, b_i),$$

where $b_i \leq a_{i+1}$ for all $i = 1, \cdots, n-1$, for $n \geq 2$. Prove that there exists a point $\xi \in (a,b)$ such that $f^{(n-1)}(\xi) = 0$.

> **Solution:** First, we prove that there exists $n-1$ points such that $f'(x) = 0$. In fact, for all $i = 1, \cdots, n-1$, by continuity, we have $f(b_i) = f(a_{i+1})$. In the case of $b_i < a_{i+1}$, by mean value theorem, there exists a point $x_i \in (b_i, a_{i+1})$ such that $f'(x_i) = 0$. If $b_i = a_{i+1}$, it is clear that b_i is a local maximum/minimum, therefore, $f'(x_i) = 0$ where $x_i = b_i$. We end up with $n-1$ distinct points $\{x_i\}_{i=1}^{n-1}$ such that $f'(x_i) = 0$.
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> Next, we apply mean value theorem on f' on the interval $[x_i, x_{i+1}]$, so there exists $n-2$ distinct points $\{y_i\}_{i=1}^{n-2}$ such that $f''(y_i) = 0$. Keep doing this procedure, we end up with the existence of a point ξ such that $f^{(n-1)}(\xi) = 0$.
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> There are typos in the problem marked red. Grades will be adjusted accordingly.

2. (20 points) Let $f : [a,b] \to \mathbb{R}$ be a bounded function which is discontinuous at a point $c \in (a,b)$. μ is a monotonically increasing function which is discontinuous at c as well. Is is true that f is not μ-integrable?

If you answer true, namely for all f and μ that shares discontinuity at some point, $f \notin \mathcal{R}(\mu)$, prove it.

If you answer false, construct a pair of f and μ that shares a discontinuity, and f is μ-integrable.

> **Solution:** False. Here is one example. $[a,b] = [-1,1]$, $c = 0$,
>
> $$\mu(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}, \quad f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}.$$
>
> We now verify $\int_{-1}^{1} f d\mu = 0$. It is clear that $L(f,P,\mu) \geq 0$ for all partition P. So It suffies to prove that $\inf_P U(f,P,\mu) = 0$.
>
> Given any partition P, consider a refinement partition $P^* = P \cup \{0\}$. By the definition of μ, the only nonzero $\Delta\mu_i$ is the cell to the left of zero: $[-\delta, 0]$.
>
> $$U(f,P^*,\mu) = \max_{x \in [-\delta,0]} f(x)(\mu(0) - \mu(-\delta)) = 0 \cdot 1 = 0.$$
>
> Hence, $\bar{\int} f d\mu = 0$ and therefore f is μ-integrable.

3. (20 points) Let $\{f_n\}$ be a sequence of $\mathcal{C}^\infty$ functions on a compact interval, such that for any integer $k \geq 0$, there exists an M_k such that $|f_n^{(k)}| \leq M_k$ for all x and n. Prove that there exists a subsequence which converges to a $\mathcal{C}^\infty$ function, in $\|\cdot\|_{\mathcal{C}^\infty}$ norm.

   Hint: apply Arzela-Ascoli theorem on $\{f_n^{(k)}\}$ and use diagonal argument. You can prove uniform convergences to get partial credit.

   Remark: The $\mathcal{C}^\infty$ norm is defined as

$$\|f\|_{\mathcal{C}^\infty([a,b])} = \sup_{k\geq 0} \max_{x\in[a,b]} |f^{(k)}(x)|.$$

Solution: First, we prove that $\{f_n^{(k)}\}$ are uniformly equicontinuous for all k, namely for all $\epsilon > 0$, there exists a $\delta > 0$, for all $|x-y| < \delta$ and all n, $|f_n(x) - f_n(y)| < \epsilon$. In fact, by mean value theorem, we know there exists a $\xi \in (x, y)$ such that $f_n^{(k)}(x) - f_n^{(k)}(y) = f_n^{(k+1)}(\xi)(x-y)$. By uniform boundedness, we get

$$|f_n^{(k)}(x) - f_n^{(k)}(y)| \leq |f_n^{(k+1)}(\xi)||x-y| \leq M_{k+1}\delta.$$

Pick $\delta = \epsilon/M_{k+1}$ (which doesn't depend on n), we get equi-continuity.

Together with uniform boundedness, we apply Arzela-Ascoli theorem and get for $k = 0$, there is a subsequence $\{f_{n_j^0}\} \subset \{f_n\}$ such that $f_{n_j^0}$ converges uniformly as $j \to \infty$.

Next, we apply Arzela-Ascoli theorem for $\{f'_{n_j^0}\}$ and there exists a subsequence $\{f_{n_j^1}\} \subset \{f_{n_j^0}\}$ such that $\{f'_{n_j^1}\}$ converges uniformly as $j \to \infty$. As $\{f_{n_j^1}\}$ also converges uniformly (due to it is a subsequence of $\{f_{n_j^0}\}$ which converges uniformly), we conclude that $\{f_{n_j^1}\}$ converges in $\mathcal{C}^1$ norm.

We continue to apply the argument for higher derivatives, and get subsequences $\{f_{n_j^k}\} \subset \{f_{n_j^{k-1}}\}$ such that $f_{n_j^k}$ converges in $\mathcal{C}^k$ norm.

Finally, consider the sequence (in the diagonal) $\{f_{n_k^k}\}_k$. By Cantor's diagonalization argument, we know the sequence converges in $\mathcal{C}^\infty$ norm.

4. (20 points) Let μ be a measure with a cumulative distribution function (also denoted by μ) $\mu : [a, b] \to \mathbb{R}$ which is bounded and monotonically increasing. Construct a sequence of simple functions ν_n such that the corresponding measure ν_n converges to μ in weak-$*$ topology.

   Remarks: 1. Simple function means step functions taking only finite many values. As ν_n is a cumulative distribution of a measure, it should also be an increasing function. The corresponding measure is a collection of point masses.
   2. ν_n converges to μ in weak-$*$ topology means for all $f \in \mathcal{C}([a, b])$,

$$\lim_{n\to\infty} \int_a^b f d\nu_n = \int_a^b f d\mu.$$

Solution: *There are many different constructions. Here we only show one of them.*

Given any n, let $P_n = \{x_i\}_{i=0}^n$ be an equally distributed partition in $[a, b]$, namely $x_i = a + \frac{i}{n}(b-a)$. Define ν_n as follows:

$$\nu_n(x) = \mu(x_{i-1}), \quad \forall x \in [x_{i-1}, x_i).$$

Clearly, ν_n is a simple function. For any continuous function f, it is clear that

$$\int_a^b f d\nu_n = \sum_{i=1}^n f(x_{i-1})(\Delta\nu_n)_i = \sum_{i=1}^n f(x_{i-1})(\mu(x_i) - \mu(x_{i-1})).$$

Compute the upper Riemann sum

$$U(f, P_n, \mu) = \sum_{i=1}^n \left(\max_{x\in[x_{i-1},x_i]} f(x) \right) (\mu(x_i) - \mu(x_{i-1})).$$

As f is (uniformly) continuous in $[a, b]$, there exists $\delta > 0$, such that if $|x - y| < \delta$, one has $|f(x) - f(y)| < \epsilon' := \frac{\epsilon}{2(\mu(b)-\mu(a))}$. Therefore, if $(b - a)/n < \delta$, then

$$\left| \int_a^b f d\nu_n - U(f, P_n, \mu) \right| \le \sum_{i=1}^n \epsilon'(\mu(x_i) - \mu(x_{i-1})) = \epsilon'(\mu(b) - \mu(a)) = \frac{\epsilon}{2}.$$

On the other hand, f is μ-integrable, and $\text{mesh}(P_n) = \frac{b-a}{n} \to 0$ as $n \to \infty$. Therefore, it is easy to check that there exists an N big enough such that for all $n \ge N$,

$$U(f, P_n, \mu) - L(f, P_n, \mu) < \frac{\epsilon}{2}.$$

To sum up, for any given ϵ, we pick $n = \max(\frac{b-a}{\delta}, N)$, then

$$\left| \int_a^b f d\nu_n - \int_a^b f d\mu \right| \le \left| \int_a^b f d\nu_n - U(f, P_n, \mu) \right| + \left| U(f, P_n, \mu) - \int_a^b f d\mu \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

5. (20 points) Ask yourself a question related to the material covered in the first half of the semester. Explain why it is interesting and nontrivial. Then try to answer it.

Solution: Everyone should have your own solution.