MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 3, Due on Wednesday, February 3, 2016

1. *(Newton's method)* Finish exercise 25 (a)-(e) in chapter 5 of Rundin's book.
Note: Newton's method is a very common way of finding a root of the function. The convergence is very fast, as you can see from (d). The drawback is that if the initial x_1 is far from the root, the assumptions is very restrictive (strictly monotone and uniformly twice differentiable), and general function f does not meet the requirement and convergence is not guaranteed. Below are OPTIONAL questions on this subject if you are interested in explore the method.

f). (No convergence) Take $f(x) = x^{1/3}$. Express x_{n+1} in terms of x_n. Does the sequence converge? Can you try to explain why it happens?

g). (Slow convergence) Take $f(x) = x^2$. Express x_{n+1} in terms of x_n. Does the sequence converge? Is the convergence rate quadratic, namely $|x_{n+1}| \le C|x_n|^2$? Can you explain why it happens? *Hint: In particular, one can calculate $g'(\xi) := \lim_{x\to\xi} g'(x)$. Quadratic convergence is equivalent to $g'(\xi) = 0$, which is not the case here.*

2. *(Different types of convergence for sequence of functions)* Let X be a metric space of functions (e.g. $\mathcal{C}([a,b]), \mathcal{C}^1([a,b]), \cdots$), with a corresponding norm (or metric, or distance) $\|\cdot\|_X$. Let $\{f_n\}$ be a sequence of functions in X. We call $\{f_n\}$ converges in X, if for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$, such that for all $n, m > N$, $\|f_n - f_m\|_X < \epsilon$.

a). Take $X = \mathcal{C}([a,b])$, with the norm defined by

$$\|f\|_{\mathcal{C}([a,b])} = \max_{x\in[a,b]} |f(x)|.$$

Prove that $\{f_n\}$ converges in $\mathcal{C}([a,b])$ is equivalent to $\{f_n\}$ converges uniformly.

b). Take $X = \mathcal{C}^1([a,b])$, with the norm defined by

$$\|f\|_{\mathcal{C}^1([a,b])} = \|f\|_{\mathcal{C}([a,b])} + \max_{x\in[a,b]} |f'(x)|.$$

Prove that if $\{f_n\}$ converges in $\mathcal{C}^1([a,b])$, then $\{f_n\}$ converges in $\mathcal{C}([a,b])$.

c). Prove that convergence in $\mathcal{C}([a,b])$ does not imply convergence in $\mathcal{C}^1([a,b])$. (This says convergence in $\mathcal{C}^1([a,b])$ is stronger.) *Hint: consider the following sequence of functions*

$$f_n(x) = \begin{cases} |x| & |x| \ge \frac{1}{n} \\ \frac{n^2x^2+1}{2n} & |x| < \frac{1}{n} \end{cases}$$

We check that $f_n \in \mathcal{C}^1([a,b])$ (in particular at $|x| = \frac{1}{n}$). Verify that $\{f_n\}$ converges in $\mathcal{C}([a,b])$ but does not converge in $\mathcal{C}^1([a,b])$.

d). (Optional) One can also define convergence in $\mathcal{C}^{0,\alpha}([a,b])$ with $\alpha \in [0,1]$, where the Hölder norm is defined as

$$\|f\|_{\mathcal{C}^{0,\alpha}([a,b])} = \|f\|_{\mathcal{C}([a,b])} + \max_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Prove that convergence in $\mathcal{C}^{0,\alpha}([a,b])$ is weaker than convergence in $\mathcal{C}^1([a,b])$ but stronger than convergence in $\mathcal{C}([a,b])$.