MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 10, Due on Wednesday, April 13, 2016

1. *(Vector and matrix norms)* The $p-$norm for $\mathbb{R}^n$ is defined as

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p} \quad \text{for } p \in [1,\infty), \quad \text{and } \|v\|_\infty = \max_{1\le i\le n} |v_i|.$$

a). Prove that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Namely, prove that there exists constant c, C such that for all $v \in \mathbb{R}_n$

$$c\|v\|_2 \le \|v\|_\infty \le C\|v\|_2,$$

where the constant can depend on the dimension n.

b). (Optional) Prove that $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent for any $1 \le p, q \le \infty$.

Let A be an n-by-n matrix. The $p-$norm for A is defined as

$$\|A\|_p = \sup_{v\neq 0} \frac{\|Av\|_p}{\|v\|_p}.$$

c). Prove that the matrix norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. (Same arguments gives equivalency for all $p-$norms.)

2. *(Fréchet derivative versus Gâteaux derivative)* Let f be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

a). Prove that if f is Fréchet differentiable, then it is Gâteaux differentiable.

b). Construct a mapping f which is Gâteaux differentiable but not Fréchet differentiable.

3. *(Differentiability implies continuity)* Suppose f is a real-valued function defined in an open set $E \in \mathbb{R}^n$.

a). Prove that if f is Fréchet differentiable in E, then f is continuous in E.

b). If we only assume that all partial derivatives $\partial_{x_1} f, \cdots, \partial_{x_n} f$ are bounded in E. Is f continuous in E? If yes, prove it. If no, provide a counter example.