# MATH302 Elements of Analysis **Spring 2018**

**Homework 5, Due on Tuesday, March 20, 2018**

For all the proofs, write *in details* using standard mathematical language.

**1.** If $S$ is a set in $\mathbb{R}$ that is bounded above, show that $\sup S$ is either an element of $S$ or is a cluster point of $S$.

**2.** Let $M$ be a metric space. Prove that $S$ is a dense subset of $M$ if and only if $S' = M$.

**3.** *(Closure)* Let $M$ be a metric space. A closure of a set $S \subseteq M$, denoted by $\bar{S}$, is defined as the union of the set and its derived set

$$\bar{S} = S \cup S'.$$

- **a)**. Prove that for any set $S$, its closure $\bar{S}$ is a closed set.
- **b)**. Prove that if $S$ is closed, then $\bar{S} = S$. *(This will directly imply that $\bar{\bar{S}} = \bar{S}$.)*
- **c)**. Prove that $\bar{S}$ is the smallest closed set that contains $S$. Namely,

$$\forall \text{ closed set } U \in M \ (S \subseteq U \Rightarrow \bar{S} \subseteq U).$$

**4.** Show that the Bolzano-Weierstrass theorem fails in the field in *any* ordered field that is not Archimedean.

**5.** Finish problem 9 in exercise 7.6 of the textbook.

- **a)**. Show that a bounded set having exactly one cluster point is denumerable.
- **b)**. Show that the assumption in part a) that $S$ is bounded is unnecessary.
- **c)**. Show that a set haing finitely many cluster points is denumeratble.
- **d)**. (Bonus) Is a set having denumerable many cluster points necessarily denumerable? Explain.
- **e)**. Is a set having uncountable many cluster points necessarily uncountable? Explain.