MATH302 Elements of Analysis **Spring 2018**

Homework 4, Due on Tuesday, Febuary 27, 2018

For all the proofs, write *in details* using standard mathematical language.

1. *($\mathbb{Q}$ does not have the least upper bound property)* Let $S = \{x \in \mathbb{Q} \;:\; x^2 < 2\}$ be a subset of the ordered field $\mathbb{Q}$. Show that S does not have a least upper bound.

Proof: Suppose S has a least upper bound $u = \sup S$.

- **a)**. If $u^2 = 2$, find a contradiction.
- **b)**. If $u^2 > 2$, find a contradiction.
- **c)**. If $u^2 < 2$, find a contradiction.

Therefore, S does not have a least upper bound. $\square$

2. *(Density)* Finish problem 12 in page 93 of the textbook.

- **a)**. If D is dense in the real line and $D \subseteq S$, show that S is dense in the real line.
- **b)**. Show that if S is dense in the real line and a finite number of points are removed from S, the resulting set is also dense in the real line.
- **c)**. Does b) necessarily remain true if the set that is removed is infinite?

Remark: 1. there is a part d) in the textbook. But it is a direct consequence of part b);
2. All statements can be extended to metric spaces.

3. *(Non-Archimedean ordered field)* Let F be a non-Archimedean ordered field, and let

$$U = \{x \;:\; x \text{ is an upper bound of } \mathbb{N}.\}$$

- **a)**. Prove that $U \neq \emptyset$.
- **b)**. Prove that U has a lower bound, but has no infinum (least lower bound).

Remark: In class, we proved least upper bound property implies Archimedean. Here, we show non-Archimedean implies violation of least upper bound. This provides an alternative proof through contradiction.

- **c)**. (Optional, no submission) Read the Wikipedia page on "hyperreal number". It is a typical example of non-Archimedean field. Check that the lesat upper bound property does not hold.