MATH302 Elements of Analysis **Spring 2018**

Homework 3, Due on Tuesday, Febuary 20, 2018

For all the proofs, write *in details* using standard mathematical language.

1. *(Non-uniqueness of positive set)* Let $\mathbb{Q}(\sqrt{2})$ be the set defined as

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}.$$

a). Prove that $(\mathbb{Q}(\sqrt{2}), +, \times)$ is a field.

You only need to check rule (0) closure under + and ×, and rule (8) existence of multiplicative inverse. All other rules are trivial.

b). Since $\mathbb{Q}(\sqrt{2})$ is a subset of $\mathbb{R}$, show that $(\mathbb{Q}(\sqrt{2}), P_1)$ is an ordered field, with the positive set P_1 defined as

$$P_1 = \{a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2}) : a + b\sqrt{2} \text{ is a positive real number}\}.$$

c). Prove that $(\mathbb{Q}(\sqrt{2}), P_2)$ is also an ordered field, with the positive set P_2 defined as

$$P_2 = \{a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2}) : a - b\sqrt{2} \text{ is a positive real number}\}.$$

2. *(Uniqueness of positive set under additional assumptions)* Suppose F is a field with a positive set P such that

$$(i)\ x, y \in P \Rightarrow x + y, xy \in P;$$
$$(ii)\ \text{Exactly one of the three holds: } x \in P, -x \in P, x = 0;$$
$$(iii)\ \forall\ x \in P\ \exists\ y \in F\ (x = y^2).$$

Prove that the positive set P is unique.
Note: condition (iii) is satisfied for $\mathbb{R}$. Therefore, $\mathbb{R}$ has a unique order.
Hint: Let $A = \{x^2 : x \in F\}$. Prove that $P = A$ (by showing $P \subseteq A$ and $P \supseteq A$).

3. Prove the triangle inequality (Theorem 4.16 in the textbook).

4. Consider the space $\mathbb{R}^2 = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ with the following metric

$$d_\infty(x, y) = |x - y|_\infty := \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

a). Prove that d_∞ is a metric.

b). What is the unit ball in $(\mathbb{R}^2, |\cdot|_\infty)$? It is defined as $B(0,1) = \{x \in \mathbb{R}^2 : |x|_\infty < 1\}$. Draw the ball in the $\mathbb{R}^2$ plane.

c). (Bonus) Prove that $B(0, 1)$ is an open set in $(\mathbb{R}^2, |\cdot|_\infty)$.

5. Finish excersise 4 in page 72 of the textbook.

a). If $I = (a, b)$ and $J = (c, d)$ are open intervals, show that $I \subseteq J$ if and only if $a \geq c$ and $b \leq d$.

b). Does this result change if the intervals are not open?