MATH302 Elements of Analysis **Spring 2018**

Homework 2, Due on Tuesday, Febuary 6, 2018

For all the proofs, write *in details* using standard mathematical language.

1. Let $P(x, y)$ be an open statement with variables x and y.

a). Prove that the statement

$$\exists\ x\ \forall\ y\ (P(x, y) \text{ is true})$$

implies the statement

$$\forall\ y\ \exists\ x\ (P(x, y) \text{ is true}).$$

b). Show that the second statement does not imply the first statement.

2. Let $P(\mathbb{N})$ be the *power set* of $\mathbb{N}$, which is the collection of all subsets of $\mathbb{N}$.

a). Prove that $P(\mathbb{N})$ is an uncountable set.
Hint: to prove $P(\mathbb{N})$ does not have the same cardinality as $\mathbb{N}$, one can use the Cantor's diagonalization argument.

b). (Bonus) Prove that $P(P(\mathbb{N}))$ is an uncountable set which does not have the same cardinality as $P(\mathbb{N})$.
Remark: this is an example that uncountable sets might have different cardinality. The proof should be seminar to part (a). But be careful that you can not write down the elements in $P(\mathbb{N})$ as $(x_1, x_2, \cdots)$ as $P(\mathbb{N})$ does not have the same cardinality as $\mathbb{N}$.

3. Let $(\mathbf{F}, +, \times)$ be a field.

a). Show that F has at most one additive identity.
Hint: you will prove if x and y are both additive identities, then $x = y$.

b). Denote 0 be the additive identity of $\mathbf{F}$. Prove that $\forall\ x \in \mathbf{F}\ (0 \times x = 0)$.
Note: you can only take the rules in the definition of field for granted.

4. Finish problem 15 in page 61 of the textbook. The problem states as follows.

a). Suppose that S is a subset of $\mathbb{N}$ with the properties:

$$\begin{aligned} &(i)\ \ 2^n \in S\ \ \forall\ n \in \mathbb{N} \\ \text{and}\quad &(ii)\ \ \text{If } k \in S \text{ and } k > 1,\ \text{then } k - 1 \in S. \end{aligned}$$

Show that $S = \mathbb{N}$.

b). Condition (i) may be phrased "$P = \{2^n\ :\ n \in \mathbb{N}\} \subset S$". State a *condition* (as opposed to a *description of*) the set P that would yield the same result.

c). Could the condition "$k - 1 \in S$" be replaced by "$k - 2 \in S$" and keep the result?

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