MATH302 Element of Analysis **Spring 2018**

Final Exam Name: ____________________

Instructions: *Please read carefully!!*

- Print out the exam in *single-sided* paper and staple on the side. Write your answer directly after the problem on the same page. You may continue on the back of the paper if necessary.
- Show all work clearly and in order. For all the proofs, write *in details* using standard mathematical language.
- The exam is limited to *3 consecutive hours* of your choice. Write starting/ending time below.
- You are allowed to use the textbook and your own (handwritten or typed) lecture notes during the exam. Other resources or communication with others are not allowed.
- You are required to sign the *honor pledge* below.
- Submit your exam on Monday April 30, between 10am and 1pm, to 426 HBH building. For alternative submission time, email request has to be made before April 26.

Honor Pledge: The Rice University Honor Pledge reads:

> "On my honor, I have neither given nor received any unauthorized aid on this exam."

Please write the exact wording of the Pledge, following by your signature, in the space below:

Pledge: ____________________

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Your Signature: ____________________

Starting time:____________________ **Ending time:**____________________

Grade box: For grading use. Please leave it blank.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	20	20	20	20	20	20	20	20	20	20	200
Score:											

Good Luck

1. (20 points) Determine whether the following statements are true or false. You do NOT need to explain why.

   (4 points for each correct answer, -1 point for each wrong answer, 0 point if you leave it blank.)

   (a) For any statements A and B, $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$.

   (b) If A is a proper subset of B, then A and B can not have the same cardinality.

   (c) Suppose F is an ordered field that satisfies the Archimedean property. Then, every bounded non-empty subset in F has a least upper bound.

   (d) Let $\{U_n\}_{n=1}^{\infty}$ be an infinite collection of closed sets in metric space S. Then, their intersection $\cap_n U_n$ is also a closed set in S.

   (e) Let f be a mapping from $\mathbb{R}$ to $\mathbb{R}$, and S is a dense subset of $\mathbb{R}$. Then, $f^{-1}(S)$ (the pre-image of S under f) is also a dense subset of $\mathbb{R}$.

(a)	
(b)	
(c)	
(d)	
(e)	

2. (20 points) Show that for any $q \in (-1,1)$, $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} q^j = \frac{q - q^{n+1}}{1-q}.$$

Note: this result leads to the formula for geometric series:

$$\sum_{j=m}^{n} q^j = q^{m-1} \sum_{j=1}^{n-m+1} q^j = \frac{q^m(1-q^{n-m+1})}{1-q}, \quad \forall\, n \geq m.$$

3. (20 points) "THEOREM": Every nonempty set is a neighborhood of at least one of its point.

   Proof: Let A be a nonempty set. Let $u = \sup A$. Since $u = \sup A$, there exists a number $\epsilon > 0$ so that $(u - \epsilon, u) \subseteq A$. Let $x = u - \epsilon/2$. Now the interval $(x - \epsilon/4, x + \epsilon/4)$ is contained in the interval $(u - \epsilon, u)$, and so $(x - \epsilon/4, x + \epsilon/4)$ is contained in A. Thus A is a neighborhood of x.

   (a) The proof contains at least two serious errors. Find them and explain why they are errors.

   (b) Is the "THEOREM" correct? If so, prove it. If not, find a counterexample.

4. (20 points) Let S be a dense set in $\mathbb{R}$. Determine whether the following statements are true or false. If true, prove the statement. If false, find a counter example.

   (a) If S is open, then $S = \mathbb{R}$.

   (b) If S is closed, then $S = \mathbb{R}$.

5. (20 points) (a) Suppose S is a subset of $\mathbb{R}$ with the property that $S \cap [-n, n]$ is finite for each $n \in \mathbb{N}$. Show that S is countable.

(b) Show that every uncountable subset of $\mathbb{R}$ has a cluster point.

6. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall f is *pointwise continuous* in $\mathbb{R}$ if

$$\forall\ \epsilon > 0\ \forall\ a \in \mathbb{R}\ \exists\ \delta > 0\ \forall\ x \in (a - \delta, a + \delta)\ (|f(x) - f(a)| < \epsilon).$$

f is *uniform continuous* in $\mathbb{R}$ if

$$\forall\ \epsilon > 0\ \exists\ \delta > 0\ \forall\ a \in \mathbb{R}\ \forall\ x \in (a - \delta, a + \delta)\ (|f(x) - f(a)| < \epsilon).$$

(a) Is $f(x) = x^2$ pointwise continuous in $\mathbb{R}$? Prove or disprove it.

(b) Is $f(x) = x^2$ uniform continuous in $\mathbb{R}$? Prove or disprove it.

7. (20 points) Consider the function

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}.$$

(a) Prove that f is continuous at $a = 0$.

(b) Prove that f is discontinuous at any point $a \neq 0$.

Note: you can directly use the fact that both $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$.

8. (20 points) Suppose (x_n) is a bounded sequence such that every subsequence of (x_n) has a subsequence that converges to L. Show that (x_n) converges to L.

9. (20 points) Let (x_n) be a sequence defined as

$$x_n = \sum_{j=1}^{n} \frac{1}{j^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3}.$$

(a) Show that for any $n \in \mathbb{N}$, $x_n \leq y_n$ where

$$y_n = \frac{3}{2} - \frac{1}{2n^2}.$$

(b) Show that (x_n) is bounded.

(c) Show that (x_n) converges.

10. (20 points) Let (x_n) be a recursively defined sequence

$$x_1 = 1, \quad x_2 = 2, \quad x_{n+2} = \frac{1}{3}x_{n+1} + \frac{2}{3}x_n, \quad \forall\, n \in \mathbb{N}.$$

(a) Prove that the sequence is Cauchy. (Hence, it converges.)
Note: you can directly use the formula for geometric series in problem 2.

(b) (Bonus 5 points) Find the limit.