MATH141(0332) Calculus II

Quiz 5 - 6, October 16 - 21, 2008 **Solution of the quiz**

Show all work clearly and in order **in separated papers**, and circle your final answers.
This is a take-home quiz. Please hand in your solution in the discussion on October 21(Tuesday).
This quiz is worth 25 points. 20 of them will be counted to your final score.

1. (*6 points*) Let a, b be non-zero constants. Verify the following formula.

$$\int e^{ax} sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a \cdot sin(bx) - b \cdot cos(bx)) + C$$

Hint: Use the strategy of integration by parts to $\int e^{ax} sin(bx)dx$ twice. Compare the result and the original function.
Solution:
We do integration by parts to $\int e^{ax} sin(bx)dx$, by letting $F'(x) = e^{ax}, G(x) = sin(bx)$. So we've got $F(x) = \frac{1}{a}e^{ax}$.

$$\int e^{ax} sin(bx)dx = \frac{1}{a}e^{ax} sin(bx) - \int \frac{1}{a}e^{ax} \cdot bcos(bx)dx = \frac{1}{a}e^{ax} sin(bx) - \frac{b}{a}\int e^{ax} cos(bx)dx$$

Then, we do integration by parts to $\int e^{ax} cos(bx)dx$, by letting $F'(x) = e^{ax}, G(x) = cos(bx)$. This time we still have $F(x) = \frac{1}{a}e^{ax}$.

$$\int e^{ax} cos(bx)dx = \frac{1}{a}e^{ax} cos(bx) - \int \frac{1}{a}e^{ax} \cdot b(-sin(bx))dx = \frac{1}{a}e^{ax} cos(bx) + \frac{b}{a}\int e^{ax} sin(bx)dx$$

Plug this equation into the first equation, we have

$$\begin{aligned}\int e^{ax} sin(bx)dx &= \frac{1}{a}e^{ax} sin(bx) - \frac{b}{a}\left(\frac{1}{a}e^{ax} cos(bx) + \frac{b}{a}\int e^{ax} sin(bx)dx\right) \\ &= \frac{1}{a}e^{ax} sin(bx) - \frac{b}{a^2}e^{ax} cos(bx) - \frac{b^2}{a^2}\int e^{ax} sin(bx)dx\end{aligned}$$

Put the last term to the left hand side, we get

$$\left(1 + \frac{b^2}{a^2}\right)\int e^{ax} sin(bx)dx = \frac{1}{a}e^{ax} sin(bx) - \frac{b}{a^2}e^{ax} cos(bx) + C'$$

Multiply $\frac{a^2}{a^2+b^2}$ in both sides. we get

$$\begin{aligned}\int e^{ax} sin(bx)dx &= \frac{a^2}{a^2+b^2}\left(\frac{1}{a}e^{ax} sin(bx) - \frac{b}{a^2}e^{ax} cos(bx)\right) + C \\ &= \frac{e^{ax}}{a^2+b^2}(a \cdot sin(bx) - b \cdot cos(bx)) + C\end{aligned}$$

2. (*6 points*) Let m and n be positive integers (both m and n are constants). Prove that

$$\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \begin{cases} 0 & if\ m \neq n \\ \pi & if\ m = n \end{cases}$$

Hint: We have the following formula

$$sin(a) \cdot sin(b) = \frac{1}{2}[-cos(a+b) + cos(a-b)].$$

Take $a = mx, b = nx$. Apply the formula to the intergral and get the answer.
Solution:
Using the formula given in the hint part. Set $a = mx, b = nx$, we have

$$\begin{aligned} \int_{-\pi}^{\pi} sin(mx)sin(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi}[-cos((m+n)x) + cos((m-n)x)]dx \\ &= \frac{1}{2}\left[-\int_{-\pi}^{\pi} cos((m+n)x)dx + \int_{-\pi}^{\pi} cos((m-n)x)dx\right] \end{aligned}$$

For the first part

$$\int_{-\pi}^{\pi} cos((m+n)x)dx = \left[\frac{1}{m+n}sin((m+n)x)\right]_{-\pi}^{\pi} = \frac{1}{m+n}[sin((m+n)\pi) - sin((m+n)(-\pi))]$$

Notice that for every integer k, $sin(kx) = 0$. So $sin((m+n)\pi) = sin((m+n)(-\pi)) = 0$.

$$\int_{-\pi}^{\pi} cos((m+n)x)dx = 0$$

For the second part, if $m \neq n$,

$$\int_{-\pi}^{\pi} cos((m-n)x)dx = \left[\frac{1}{m-n}sin((m-n)x)\right]_{-\pi}^{\pi} = \frac{1}{m-n}[sin((m-n)\pi) - sin((m-n)(-\pi))] = 0$$

If $m = n$, the former way does not work because $\frac{1}{m-n}$ is not defined.

$$\int_{-\pi}^{\pi} cos((m-n)x)dx = \int_{-\pi}^{\pi} cos(0)dx = \int_{-\pi}^{\pi} 1dx = 2\pi$$

So,

$$\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \begin{cases} \frac{1}{2}(-0+0) = 0 & if\ m \neq n \\ \frac{1}{2}(-0+2\pi) = \pi & if\ m = n \end{cases}$$

3. (*7 points*) Let a, b, c be positive constants. Solve the volume of the solid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

Hint: Recall disc mothod in Chapter 6.
Step 1: Determine x-axis.
Step 2: Find $A(x)$. Fixed x, we have

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 - \frac{x^2}{a^2}.$$

It is an ellipse on y-z coordinates. You should find its area. To do this, you need to write z in terms of y. Notice, a, b, c, x are all constants now. When solving the integral, Section 8.3 will help.
Step 3: $V = \int_{-a}^{a} A(x)dx$.

Solution:
We set the x-axis to be the x-axis of the function of the solid.
So if we fixed $x = x_0$, We have the function of the area perpendicular to x-axis at the point x_0 to be

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 - \frac{x_0^2}{a^2}.$$

in terms of y and z.
Let $r = \sqrt{1 - \frac{x_0^2}{a^2}}$. r is a constant. We have the following function in $y - z$ coordinates

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} \leq r^2.$$

It is an ellipse. I first put z in terms of y to solve the upper half of the image.

$$z(y) = c\sqrt{r^2 - \frac{y^2}{b^2}} = \frac{c}{b}\sqrt{b^2r^2 - y^2}$$

To solve the integral, first we determine the interval of y. It should be from $-br$ to br. Then, we use the substitution of $y = (br)sin(u)$ (based on what we have learned in Section 8.3).

$$\begin{aligned}
\int_{-br}^{br} z(y)dy &= \frac{c}{b}\int_{-br}^{br} \sqrt{b^2r^2 - y^2}dy \\
&= \frac{c}{b}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} brcos(u) \cdot brcos(u)du \\
&= bcr^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^2(u)du \\
&= bcr^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + cos(2u)}{2}du \\
&= \frac{bcr^2}{2}\left[u + \frac{1}{2}sin(2u)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}bcr^2
\end{aligned}$$

So, the area of the ellipse should be bcr^2.

$$A(x_0) = \pi bcr^2 = \pi bc\left(1 - \frac{x_0^2}{a^2}\right) = \frac{\pi bc}{a^2}(a^2 - x_0^2)$$

At last, we do $\int A(x)dx$ to get the volume of the solid. Again, we shall use the substitution of $x = asin(u)$.

$$\begin{aligned}
V = \int_{-a}^{a} A(x)dx &= \frac{\pi bc}{a^2}\int_{-a}^{a}(a^2 - x^2)dx \\
&= \frac{\pi bc}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a} \\
&= \frac{\pi bc}{a^2}\left[(a^3 - \frac{1}{3}a^3) - (-a^3 + \frac{1}{3}a^3)\right] \\
&= \frac{\pi bc}{a^2} \cdot \frac{4}{3}a^3 \\
&= \frac{4}{3}\pi abc
\end{aligned}$$

4. (*6 points*) Solve the integral

$$\int \frac{x^3 + 3x^2 + x}{x^3 + x^2 - 2} dx$$

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Hint: We have the following equality

$$x^3 + x^2 - 2 = (x-1)(x^2 + 2x + 2)$$

Solution:

$$\int \frac{x^3 + 3x^2 + x}{x^3 + x^2 - 2} dx = \int \left(1 + \frac{2x^2 + x + 2}{x^3 + x^2 - 2}\right) dx = x + \int \frac{2x^2 + x + 2}{x^3 + x^2 - 2} dx$$

We need to solve the integral of the second part.
We want to find out constants A, B, C such that

$$\frac{2x^2 + x + 2}{x^3 + x^2 - 2} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 2x + 2}$$

Solve the right hand side

$$\begin{aligned} \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2} &= \frac{A(x^2+2x+2) + (Bx+C)(x-1)}{(x-1)(x^2+2x+2)} \\ &= \frac{(A+B)x^2 + (2A - B + C)x + (2A - C)}{x^3 + x^2 - 2} \end{aligned}$$

Compare with the left hand side, we get

$$\begin{cases} A + B &= 2 \\ 2A - B + C &= 1 \\ 2A - C &= 2 \end{cases} \Longrightarrow \begin{cases} A &= 1 \\ B &= 1 \\ C &= 0 \end{cases}$$

So, we have

$$\int \frac{2x^2 + x + 2}{x^3 + x^2 - 2} dx = \int \frac{1}{x-1} dx + \int \frac{x}{x^2 + 2x + 2} dx$$

For the first term,

$$\int \frac{1}{x-1} dx = ln|x-1| + C$$

We separate the second term to be two parts.

$$\int \frac{x}{x^2 + 2x + 2} dx = \int \frac{x+1}{x^2 + 2x + 2} dx - \int \frac{1}{x^2 + 2x + 2} dx$$

For the first part, notice that $d(x^2 + 2x + 2) = (2x + 2)dx = 2(x+1)dx$, so

$$\int \frac{x+1}{x^2 + 2x + 2} dx = \frac{1}{2} \int \frac{d(x^2 + 2x + 2)}{x^2 + 2x + 2} = \frac{1}{2} ln(x^2 + 2x + 2) + C$$

For the second part, we use the formula of 'arctan'.

$$\int \frac{1}{x^2 + 2x + 2} dx = \int \frac{1}{(x+1)^2 + 1} dx = arctan(x+1) + C$$

Put them together, we find the final integral

$$\int \frac{x^3 + 3x^2 + x}{x^3 + x^2 - 2} dx = x + ln|x-1| + \frac{1}{2} ln(x^2 + 2x + 2) - arctan(x+1) + C$$